WARING'S CONJECTURE AND A CERTAIN DIOPHANTINE EQUATION

Mike Mudge starts a new series of mathematical puzzles.

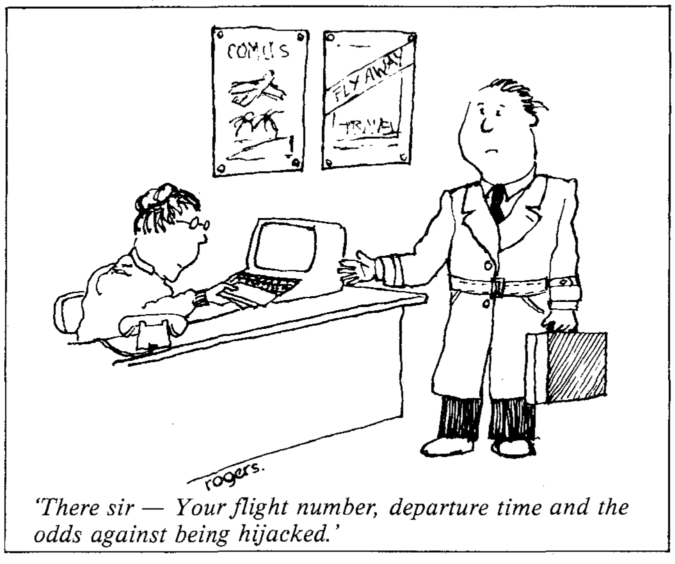
PCW has always prided itself on its ability to provide answers to that perennial question, 'what can I use my computer for?' Most of the answers, for hobbyist or enthusiast, have been of a recreational or educational nature. There is however an area of serious research which is open to (and well within the capabilities of) the mathematically-minded computer hobbyist. I refer to the field of Number Theory. Given that its subject matter — the natural numbers — is infinite, it's not surprising that there remain huge numbers of unsolved problems; moreover, the scope for discovering new problems of interest is limitless.

From this issue on, we will be publishing a monthly column by Mike Mudge BSc FIMA FBCS of the University of Aston, in which he sets problems in Number Theory (with explanatory background) and awards a monthly prize of £10 for the best submission. These problems will not be simple puzzles but genuine research projects in Number Theory which are capable of investigation using a personal computer; who knows, we may even arrive at some important results? Don't be discouraged from trying them if you're not a professional mathematician; if you've enjoyed our Leisure Lines puzzles or the Manhunt Competition you might find that you're a budding number theorist already! — *Dick Pountain.*

Waring's conjecture

The integers consist of 0, ±1, ±2, ±3, . . . eg, +1234 or -201379. When k denotes a positive integer, the product of k-factors each equal to x is written x^k, eg, $7^3 = 7.7.7 = 343$.

A Diophantine equation is one which is to be solved using integers only, the first writer to study such equations being Diophantus of Alexandria (c AD 250). For example $x^2 + y^2 = z^2$ regarded as a Diophantine equation has among its solutions x = 3, y = 4, z = 5 and x = 5, y = 12, z = 13 – each being the integer length sides of a Pythagorean (or right-angled) triangle.



'There sir — Your flight number, departure time and the odds against being hijacked.'

In 1770, in a text entitled *Meditationes Algebraicae*, the mathematician Edward Waring wrote (in Latin): 'Every positive integer can be expressed as the sum of at most, g(k), k^{th} powers of the positive integers, where g(k) depends only on k, not on the number being represented.'

Special Cases k = 2,3,6.

It was proved by Lagrange in 1770 that every positive integer can be expressed as the sum of at most *four squares.* Other theoretical results to date include:
i. every positive integer can be expressed as the sum of at most *nine cubes.*
ii. every positive integer can be expressed as the sum of at most 73 *sixth powers.*

Problem

Given the Diophantine equation $x^3 + y^3 + 2z^3 = k$ where k is a known positive integer, what are the (integer) solutions for x, y and z?

Historical note

In 1969 M Lal, W Russell and W J Blundon (*Math Comp* Vol 23) reported a calculation which was originally programmed in Fortran and subsequently in assembler (showing an acceleration factor of 15x) for an IBM 1620; after 1000 hours at low priority they had considered $-10^5 < x,y,z < 10^5$ and all k between 1 and 999. Their computation revealed the results

$(-133)^3$	+	$(-46)^3$	+	$2(107)^3$	=	113
$(-602)^3$	+	$(450)^3$	+	$2(309)^3$	=	190
$(-79)^3$	+	$(126)^3$	+	$2(-91)^3$	=	195

omitted by the previous writer, Makowski, in 1959.

However they failed to find any values of x,y,z corresponding to the following 19 k values less than 1000:

76	148	183	230	253
356	418	428	445	482
491	519	580	671	734
788	923	931	967	

Submit a program which generates these numbers and attempt to eliminate some of them by an extended x,y,z search or otherwise. Alternatively extend the k-range, hence possibly adding to the above list.

All submissions should include program listings, hardware description, run times and output; they will be judged for accuracy, originality and efficiency (not necessarily in that order) and I shall award a suitable prize to the 'best' entry.

Submissions to: M R Mudge BSc FIMA FBCS, Room 560/A, Department of Mathematics, The University of Aston in Birmingham, Gosta Green, Birmingham B4 7ET.

Note: Submissions will only be returned if suitable stamped addressed envelopes are included.