NUMBERS

Mathematical mind-benders from Mike Mudge

The notion of a congruent number has been familiar to some mathematicians for at least a thousand years. The defining algorithm leading to the construction of such numbers is easily illustrated by a simple example:

(i) Take three squares, $a^2<b^2<c^2$, which have a common difference, D (we say that they are in arithmetic progression): for example, $a^2=1^2=1$, $b^2=5^2=25$, $c^2=7^2=49$, here D=49−25=25−1=24.

(ii) Take the common difference, D, and write it in the form Nd^2, where N is square free: for example, $D=24=2^2\times 6$.

(iii) N is then a *congruent number* (6 is such a number).

The congruent numbers 5, 6, 14, 15, 21, 30, 34, 65, 70, 110, 154, 190, 210, 221, 231, 286, 330, 390, 429, and 546, together with 10 more less than a thousand, were given on an Arab manuscript *c*900AD.

However, it was left to L Bastien in 1915 to establish the congruent number 101, for which the smallest associated integers are:

a=16281243707272699996961,
b=20152424629497600001961
c=2339148435306225006961,
d=118171431852779451900

Algebraic formalism We are solving $b^2 - a^2 = c^2 - b^2 = Nd^2$ and the problem is to discover which values of N are permissible.

Since $2b^2 = c^2 + a^2$ then $(2b)^2 = (c+a)^2 + (c-a)^2$, now writing $Nd^2=uv(u+v)(u-v)$ where u is even, v is odd and u and v have no common factor provides the starting point for much of the computation and theoretical study that has taken place. A summary of known results including 198 congruent and 135 non-congruent numbers less than a thousand is given by R Alter and T B Curtz (*Math Comp* vol 28, 1974, pp303).

It is now appropriate to acknowledge the assistance and advice of Robin Merson, one of this column's regular readers. He is responsible for the following new approach and also suggested the inclusion of this topic in the Numbers column.

We give the detailed algebra for odd N; the word of encouragement being that following through this analysis on your micro should readily yield the non-congruent number 105 which is not well known in the current literature: Set $u=4n_1p^2$, $v=n_2q^2$, $u+v=n_3r^2$, $u-v=n_4es^2$, where $n_1n_2n_3n_4=N$ and $e=+1$ or $e=-1$.

So, we have the four equations $4n_1p^2+n_2q^2=n_3r^2$, $4n_1p^2 - n_2q^2 = n_4es^2$, $8n_1p^2 = n_3r^2 + n_4es^2$, $2n_2q^2 = n_3r^2 - n_4es^2$.

Taking congruences modulo 8, it is seen that n_2-n_3 is divisible by 4, and n_3+en_4 is divisible by 8. These are linear restrictions. Taking congruences modulo each n in turn, we obtain quadratic restrictions: for example, n_2n_3 is a quadratic residue of n_1 by which we mean that there is a value of x for which $x^2 - n_2n_3$ is divisible by n_1. There are twelve such restrictions which are not completely independent.

Readers are invited to submit a program or programs to determine congruent and/or non-congruent numbers. They may reasonably restrict the search to numbers less than a thousand, although this is not obligatory.

Try the effect of making two or three of u,v,u+v, u−v, perfect squares; this is equivalent to having two or three of n_1, n_2, n_3, n_4 unity. To obtain some non-congruent numbers take a square free N, factorise it, allocate its factors in all possible ways to n_1, n_2, n_3, n_4, testing each allocation for the restrictions. If all ways fail, then N is non-congruent.

Submissions should include program listings, hardware description, run times and output. These will be judged for accuracy, originality and efficiency (not necessarily in that order) and a prize will be awarded to the 'best' entry received by 1 April 1985.

Please address entries to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, near Wolverhampton, Staffs WV4 5NF. Tel: (0902) 892141.

Tree-like structures, August 1984

The 3× + 1 Problem, also known as Collatz', Kakutani's, Syracuse and Ulam's Problem or Hasse's Algorithm attracted considerable response. Volume one, number eight, June 1984 of *Quarch* (Archimedeans, Cambridge University Mathematical Society Newsletter) contains a detailed report on the state of the art; appropriate since this problem first prompted the newsletter in April 1980.

J C Lagarias, *The 3× + 1 problem and its generalisations,* American Mathematical Monthly, 1984/5 surveys the large amount of work done in this field. Responses from *PCW* readers included reference to R E Crandall's *Math Comp*, Vol 32, where the sequence $a_{n+1}=\tfrac{1}{2}a_n$, a_n even and $a_{n+1} = da_n + 1$, a_n odd is discussed. For d=1 this is trivial, if d=5,181 or 1093 the recurrence does *not* reduce to a cycle involving 1 for all a_0. For d=7, $a_0=3$ the behaviour is unknown.

Dr D Fisher supplied a one-line statement of the problem:

$$x_n + 1 = \tfrac{1}{2}\left(x_n 3^{(1-\cos\pi x_n)/2} + (1-\cos\pi x_n)/2\right)$$

as a special case of a 'chaotic iteration', $x_n + 1 = Lx_n(1-x_n)$. No attempt was made to use any graphics to display tree-like structures; this topic will be returned to in a later problem.

This month's worthy prizewinner is Fred Salt of 'The Paddock', Flanders Road, Llantwit Major, South Glamorgan CF6 9RL. The work was carried out in Apple Pascal on a U200 (Apple compatible) computer with 48k RAM + a 16k language card; the results were displayed using an Epson MX80 FT III printer. A suitable prize will be sent to 'The Paddock'. Further enquiries relating to this work should be directed either to Fred Salt or myself.

Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Expanded reviews of previous problems together with, subject to the approval of the contributor, copies of detailed programs from the prize-winning entry may also be requested.

END

LEISURE LINES

Brain-teasers courtesy of J J Clessa

Quickie

Put two pennies on the table. If you keep one fixed and roll the other around its edge, always touching, how many revolutions will it make? One, you might think? Try it and see.

Prize Puzzle

I bought a book the other day — it was an exciting mystery story. But I discovered, after I was well into it, that a whole section of pages was missing. I calculated that the total of the page numbers on the missing sheets was 2567. What were the missing pages?

Answers, on postcards only, to: *PCW* Prize Puzzle, January 1985, Leisure Lines, 62 Oxford Street, London W1. Entries to arrive not later than 31 January 1985.

September Prize Puzzle

Over 200 entries were received for this not too difficult logic problem. In fact, the most difficult bit for a Greek entrant was knowing what a 'postcard' was so that he could send in his (correct)