NUMBERS COUNT

Calling all graphics enthusiasts. This month Mike Mudge examines the overlap between geometry and number theory.

We draw on the entire plane squares of unit size, like those found on graph paper; the vertices of these squares are called 'Lattice Points'. Such points have been the subject of many interesting mathematical investigations since the time of Karl Friedrich Gauss (1777-1855).

We give, in increasing order of difficulty, five questions relating to these lattice points together with the state of the art regarding their solu-

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tion as known to the author. Each is followed by a programming problem where it is assumed that the programmer has access to at least a minimal graphics facility, involving the ability to display lattice points, also circles having a given centre and radius together with straight lines passing through two given points.

Question 1 (due to Hugo Steinhaus, the author of the highly recommended work *Mathematical Snapshots*). For every positive integer, n, does there exist in the plane a circle having in its interior exactly 'n' lattice points?

The answer is known to be yes; however, we have to allow the coordinates of the centre to be not only non-integer but also irrational, by which we mean not of the form a/b where a and b are integers. (A Schinzel.)

Problem A. Write a computer program to count the number of lattice points within a circle having a given centre and radius. Graphical output would enhance this considerably.

Question 2 (due to J Browkin). For every positive integer, n, does there exist in the plane a square containing exactly 'n' lattice points?

The answer is again known to be yes; however, the proof is considerably more difficult than that for the circle.

Problem B. Write a computer program to count the number of lattice points within a square defined by two adjacent vertices. (Note that there are, in general, two different answers: why?)

Graphical output is again desirable.

Question 3. For every positive integer, n, does there exist a set of 'n' lattice points lying on the circumference of some circle, and such that the distance between any two of them is an integer (when expressed in terms of the mesh spacing of the lattice)?

Answered in the affirmative by W Sierpinski.

Problem C. Construct and display such sets of 'n' lattice points for n = 3,4,5 ..., together with the associated circle upon which they lie.

Question 4 (due to K Zarankiewicz, 1951). For a positive n greater than or equal to three, consider the n^2 lattice points (x,y) where x and y are positive integers less than or equal to n, denote the set of these points by R_n.

What is the smallest positive integer k (dependent of course on n, so we write k(n)) for which *each* subset R_n having k(n) points contains nine points in three different rows and three different columns?

It is known that k(4) = 14, k(5) = 21, k(6) = 27 (W Sierpinski), and that k(7) = 34 (J Brzezinski).

Problem D. Write a computer program to display the n^2 lattice points and allow the user to select k(n) of these (or to delete $n^2 - k(n)$) before determining a set of nine points satisfying the above condition.

Initially restrict the program to n = 4,5,6 and 7 but, hopefully, extend the values of k(n)!

Question 5 (due to Mazurkiewicz *c* 1914). Does there exist in the plane a set of lattice points with which every straight line in the plane has exactly two points in common?

The answer is yes and has been established using the logical tool known as the axiom of choice; however, no concrete example of such a set is known.

Problem E. How can the computer help here?

Readers are invited to submit their attempts at some (or all) of the above problems to: Mike Mudge, 'Square Acre', Stourbridge Road, Penn, Wolverhampton, Staffordshire WV4 5NF. Tel: (0902) 892141.

Submissions, which must reach me by 1 July 1986, will be judged using suitably vague criteria. A prize will be awarded for the best entry received.

Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Expanded reviews of previous problems together with, subject to the approval of the contributor, copies of detailed programs from the winning entry may also be requested. However, in the interests of efficiency, interested readers are urged to contact the prizewinner directly.

October review

Responses to the topic of continued fractions were extremely varied. In addition to the references given in *PCW* (October 1985), mathematically inclined readers should consult *Exercises in Number Theory* by DP Parent (Springer Verlag 1984, ch 9).

The future interest in applied numerical continued fractions seems likely to lie in investigating the relationship between the CF expansion of an algebraic number (that is, a number which is the root of a polynomial equation with integer coefficients) and the properties of the sequence of quasi-random numbers nx − [nx], n=1,2,3...where [nx] denotes the greatest integer not greater than nx (the computer function Entier of Int).

Thus $n\sqrt{2} - [n\sqrt{2}]$ yields values approximately .4142, .8284, .2426, .6569, .0711 and so on.

Such numbers may be used to model a Uniform Distribution over the interval 0,1 for certain simulations; that is, Monte-Carlo Techniques and, in particular, numerical integration. Extensive references are available on request.

This month's prizewinner is Richard F Tindall of 26 Poplar Close, Great Shelford, Cambridge for an extensive submission combining analytical methods with the use of a NewBrain in Basic and a TI59 calculator.

Much of Richard's work is concerned with the determination of an empirical function for the longest periods of second-degree algebraic numbers.

(See CD Patterson and HC Williams' *Some Periodic Continued Fractions with Long Periods. Mathematics of Computation* (Vol 44 No 170 pp523-532 April 1985) including the square root of 46257585588439 with period 25679652. Their paper mentions the work of GF Voronoi, *On the generalisation of the algorithm of continued fractions*, Doctoral Dissertation Warsaw 1896 in Russian (any volunteers to translate?)