NUMBERS COUNT

Mike Mudge explains the relevance of Bernoulli and Euler numbers as related to Kummer's regular primes.

The subject area of Bernoulli and Euler numbers was proposed by Albert N Debono, of 21 St Anthony Street, Rabat, Malta, who would be pleased to receive any comments or suggestions for further work in connection with this topic.

Let us begin with an optional mathematical digression:

(1) Some readers will be familiar with:

$e=1+1/1! + 1/2! + 1/3! + 1/4! \ldots \approx 2.718281828$; and

also with the exponential function:

$\exp(x)=e^x=1+x/1!+x^2/2!+x^3/3!+x^4/4! \ldots$; convergent for all values of x.

It will come as no surprise to these readers to learn that:

$F(x)=x/(e^x-1)$

can be expanded as a power series in ascending powers of x. Thus we may write:

$F(x)=B_0+B_1x/1!+B_2x^2/2!+B_3x^3/3!+B_4x^4/4! \ldots$

This expansion defines the Bernoulli Numbers B_r, for $r=0,1,2,3,4 \ldots$

(2) We define a sequence of rational numbers $B_r=N_r/D_r$, (these are fractions expressed in their lowest terms; N_r and D_r having no common factors) by $B_0=1$, $B_1=-½$ and for $n \geqslant 2$ by:

Equation 2.1 $\ldots B_0/(n!\ 0!)+B_1/((n-1)!\ 1!)+B_2/((n-2)!\ 2!)+ \ldots +B_{n-1}/(1!(n-1)!)=0$.

(where 0! is defined to be 1 and $(n-1)!=1 \times 2 \times 3 \times \ldots \times (n-1)$).

Thus $B_0/(3!\ 0!)+B_1/(2!\ 1!)+B_2/(1!\ 2!)=0$ yielding $B_2=1/6$.

It is readily seen that $B_3=B_5=B_7= \ldots = 0$ now $B_4=-1/30$, $B_6=+1/42$, $B_8=-1/30$, $B_{10}=+5/66$, $B_{12}=-691/2730$, $B_{14}=+7/6 \ldots$

$N_{36}=26315271553053477373$,

$D_{60}=56786730$ where in these last two values any reference to sign has been omitted, for convenience in typing. Clearly in B_{4n} for integer n, either N_n or D_n must have a negative sign associated with them, in order to render B_{4n} negative, as the above pattern suggests.

Euler Numbers are defined in a similar manner using the secant function, thus:

$1/\cos x=\sec x =E_0-E_2x^2/2!+E_4x^4/4! - \ldots$

or alternatively:

$(1-x^2/2!+x^4/4! - \ldots)(E_0-E_2x^2/2!+E_4x^4/4! \ldots)=1$.

It is readily seen that $E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385$. However, it is somewhat more time-consuming to obtain:

$E_{24}=15514534163557086905$.

Now Ernst Eduard Kummer (1810–1893) defined a 'regular prime', P, to be a prime number which does not divide any of the numerators of the Bernoulli Numbers up to B_{P-3}; all other primes are 'irregular'.

For example 37 is irregular since it divides $N_{32}=7709321041217=37 \times 208360028141$.

The sequence of irregular primes begins 37, 59, 67, 101, 103, 131, 149, 157; these were known to Kummer in 1874.

Subsequently E Stafford and HS Vandiver used desk calculators to determine all of the 'irregular primes' up to 617 by 1937.

The relevance of the above concepts are best demonstrated in *13 Lectures on Fermat's Last Theorem* by Paulo Ribenboim (Springer 1979).

This month's problems are:

(a) to construct a computer program (using equation 2.1 above or otherwise) incorporating a cancelling routine to determine, given a positive integer N, the members of $B_r=N_r/D_r$ for $r=0,1 \ldots N$.

(b) to use the N_r together with a table of prime numbers, or otherwise, to obtain a sequence of irregular primes; and

(c) to determine Euler Numbers together with their prime factors.

Attempts to solve these problems may be submitted to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, Near Wolverhampton, Staffordshire WV4 5NF, tel (0902) 892141 to arrive by 1 April 1987. It would be appreciated if such submissions contained a brief summary of results; together with thoughts relating to these problems, in a form suitable for future publication in *PCW*.

These submissions will be judged using suitably vague criteria, and a prize will be awarded to the 'best' contribution received.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and other computational mathematics. Particularly welcome are suggestions for future 'Numbers Count' articles; all letters will be answered in due course.

Isolated readers can be put in contact with others sharing common interests; however, greater efficiency regarding published problems should result from contacting the prize-winner directly.

Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Review: S_k sets & extensions

Readers wishing to study 'Sets in which xy+k is always a square' for x and y in the set and k given are referred to the paper of the above title by Ezra Brown in *Mathematics of Computation* (Vol 45, Number 172, October 1985, pp613–670).

That paper together with its bibliography represent the state of the art as far as is known to the author. Further information is welcome.

This month's prize-winner is Geoff Lockwood, of 254 Crystal Palace Road, East Dulwich, London SE22 9JH. Geoff programmed his Apricot F1e in Fortran to list S_k sets with elements less than any given number (up to 6000) for any value of k.

Prompted in some way by 'Numbers Count', a letter from Donald Cross, of 66 Pennsylvania Road, Exeter EX4 6DF, posed a number of Diophantine problems (as well as the issue of multi-grades). I hope to return to this topic at a later date. END

MICROCHESS

By now many of you will already know the results of the 1986 World Microcomputer Chess Championship, but several months prior to the game Kevin O'Connell had to speculate on the outcome. How did he score? Read on, and find out.

By the time you read this article, the 1986 World Microcomputer Championship will already be over. The event, which as I write is still in the future, promises to be fascinating since it features another battle between Fidelity (who has won the Championship every time it has played) and Mephisto (who shared first with Fidelity in 1984 and then took sole first in 1985), while London's own Intelligent Chess Software can