NUMBERS COUNT

Mike Mudge deals with Polygonal, Pyramidal and Figurate Numbers.

Definition I: An 'Arithmetic Sequence' is a sequence of numbers each differing from the previous one by a constant.

Definition II: A 'Gnomon' is an 'Arithmetic Sequence' beginning with 1 and having a positive integer constant difference.

Thus G_1: 1,2,3,4,5,6,7,8, ... constant difference = 1

G_2: 1,3,5,7,9,11,13, ... constant difference = 2

G_3: 1,4,7,10,13,16, ... constant difference = 3

G_4: 1,5,9,13,17,21, ... constant difference = 4

Definition III: To form the 'Polygonal Numbers' you add subsequent numbers from a Gnomon such as G_1. So, for example, generating the polygonal sequence T_1 from G_1 will work as follows:

G_1 1 2 3 4 5 6 7 8...

↓+↗ ↓+↗ ↓+↗ ↓+↗ ↓+↗ ↓+↗ ↓+↗ ↓

T_1 1 3 6 10 15 21 28 36...

This sequence is called 'Triangular numbers.'

Similarly,

G_2 yields S:1,4,9,16,25,36,49,64, 81, ... the Square Numbers

G_3 yields P:1,5,12,22,35,51,70, 92, ... the Pentagonal Numbers

G_4 yields H:1,6,15,28,45,66,91, 120, ... the Hexagonal Numbers.

Note Hypsicles gave such a definition of Polygonal Numbers around 175BC. However, he did not have a simple graphics routine to display a given Polygonal Number as an array of dots of appropriate size and shape!

Definition IV: The 'Pyramidal Numbers' are those formed by taking the sum of increasing numbers of terms of a sequence of Polygonal Numbers in the same way as above.

'I don't think he's got the hang of the laser printer.'

Thus T: yields 1,4,10,20,35,56,84,120, 165, ... the Tetrahedral Numbers

S: yields 1,5,14,30,55,91,140, 204,285, ...

P: yields 1,6,18,40,75,126,196, 288,405, ...

H: yields 1,7,22,50,95,161,252, 372,525, ...

Note Hindu Aryabhatta (476 AD) gave the formula r(r+1)(r+2)/6 for the r^{th} Tetrahedral Number. However, he did not have computer graphics available to display Pyramidal Numbers! Do you?

Definition V: The r^{th} 'Figurate Number' of order n is defined by F^r_n = (r+n−1)(r+n−2) ..., (r)/(1.2.3 ... n) which many readers will recognise as the 'Binomial Coefficient'

$^{r+n-1}C_n$;

or

$\binom{r+n-1}{n}$

readily available, for restricted n & r, on most scientific pocket calculators.

Note F^r_2 is the r^{th} Triangular Number. F^r_3 is the r^{th} Tetrahedral Number; further, Fermat called F^r_4 the r^{th} Triangulo-triangular Number.

Problems From the above general theory, a sample of specific problems follows.

(i): Which Triangular Numbers consists only of repetitions of a single digit? For example: 55.

(ii): Which square numbers are also tetrahedral? For example: 1 & 4.

(iii): Which Triangular Numbers are also pentagonal? For example: 1 & 210.

(iv): Which numbers are simultaneously triangular, pentagonal and hexagonal?

(v): Which pairs of Triangular Numbers have their sum and difference also triangular?
For example: 15 & 21.

(vi): More generally, which Polygonal Numbers of a given sequence are also Pyramidal Numbers of a given sequence? For example: 22 is a Polygonal Number of the third sequence and also a Pyramidal Number of the fourth sequence.

(vii): An abstract problem! How best to represent 'graphically' the r^{th} Figurate Number of order n? Suggestion: firstly, implement algorithms to represent 'graphically' the Polygonal Numbers and then the Pyramidals.

Readers are invited to send their thoughts, together with complete or partial attempts at the investigations of the above problems, to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, Staffordshire WV4 5NF (tel: 0902 892141) to arrive by 1 November, 1987. It would be appreciated if such submissions contained a brief summary of results obtained in a form suitable for publication in *PCW*. These submissions will be judged using subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution received by the closing date.

Please note that submissions can only be returned if a stamped addressed envelope is provided.

Review: February '87

Problem (i): due to Professor Leo Alex; a number of submissions found 250 solutions. Refer to him for a proof of the finiteness of the solution set.

Problem (ii): Not too much progress here; although as an appetiser for further work:

The sum of two fourth powers in two different ways. That is:

$S=x^4 + y^4 = z^4 + w^4$ was completely solved by Euler algebraically.

$x=a^7+a^5b^2-2a^3b^4+3a^2b^5+ab^6$

$y=a^6b-3a^5b^2-2a^4b^3+a^2b^5+b^7$

$z=a^7+a^5b^2-2a^3b^4-3a^2b^5+ab^6$

$w=a^6b+3a^5b^2-2a^4b^3+a^2b^5+b^7$.

a=1, b=2 yielding x=133, y=134, z=(−)59 & w=158 where S=635318657.

a=2, b=3 yielding x=3494, y=1623, z=(−)2338 & w=3351 where S=155974778565937.

Problem (iii): attracted considerable attention; the greatest success in

(a) listing solutions up to $7094269=168^3+133^3=189^3+70^3$.

(b) The empirical evidence up to 10^8 says no! But how do we prove this?

The prizewinner, Richard Tindall, used a combination of Basic on a NewBrain and Basic (Microsoft) on the Kaypro 2000: 'Pascal is even less suitable'. Richard, of 26 Poplar Close, Great Shelford, Cambridge, would welcome enquiries regarding the details of his algorithms.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and other computational mathematics. Particularly welcome are suggestions, either general or particular, for future Numbers Count articles; all letters will be answered in due course.

Isolated readers can be put into contact with others sharing the same interests. However, greater efficiency regarding publishing problems should result from contacting the prizewinner directly.