NUMBERS COUNT

Mike Mudge's subject matter this month goes back to a 20-year-old conference at Oxford University. Why not get stuck in, as he dives into the fascinating world of 'Integer Bases'?

It is first required to define a sequence of positive integers using a simple rule, which may not necessarily be an explicit algebraic formula. Here are some typical sequences which will be used later:

(i) $f(n) = n^2 + 1$ (2, 5, 10, 17, 26, 37, . . .)

(ii) $f(n) = n^2$ (1, 4, 9, 16, 25, 36, . . .)

(iii) Prime Numbers (2, 3, 5, 7, 11, 13, . . .)

(iv) Pseudoprimes* (2, 3, 4, 5, 6, . . .)

(*See *A Handbook of Integer Sequences* by NJ Sloane (Academic Press 1973.))

(v) Primes squared (4, 9, 25, 49, 121, 169, . . .)

(vi) Triangular Nos** (1, 3, 6, 10, 15, 21, . . .)

(vii) $f(n) = n^3$ (1, 8, 27, 64, 125, 216, . . .)

(viii) $f(n) = n^3 + 1$ (2, 9, 28, 65, 126, 217, . . .)

(*See 'Numbers Count', *PCW*, August 1987, page 210.)

Let $S \equiv (s_1, s_2, s_3, \ldots s_k, \ldots)$ denote a general sequence of positive integers; assumed to be non-terminating. P(S) consists of the set of all positive integers which can be expressed as a sum of a finite number of *distinct* terms from S. S is defined to be a 'complete sequence' if, and only if, all sufficiently large integers belong to P(S). That is, there must be a largest integer which does *not* belong to P(S). This integer is called 'the threshold of completeness' of S and is denoted by T(S). For example, for the sequence defined as (i) above, $f(n) = n^2 + 1$, the largest integer which cannot be expressed as a sum of distinct elements of S is 51, we write T(S) = 51.

The corresponding numbers for sequences (ii) . . . (viii) are 128, 6, 1, 17163, 33, 12758 and 8293.

Essential completeness Given a general non-terminating sequence of positive integers S (of which eight particular examples are listed above) we now define associated truncated sequences $S_r \equiv (s_r, s_{r+1}, s_{r+2}, \ldots s_k, \ldots)$ where the r^{th} truncated sequence associated with S is formed by omitting the first r — 1 elements from S. Thus, for example (i) alongside $S_3 \equiv (10, 17, 26, \ldots)$ and it is found that the threshold of completeness for this is $T(S_3) = 255$.

A sequence S is defined to be 'essentially complete' if, and only if, all of its associated truncated sequences S_r are complete.

Theoretical importance. The theoretical importance of this work is centred upon the ratio $T(S_r)/s_{r-1}$: that is, the ratio of the threshold of completeness of the r^{th} truncated sequence to the largest term omitted in its formation.

There is a conjecture that this ratio does not exceed 3 for the sequence (iii) of primes, nor does it exceed 5 for the sequence (ii) generated by n^2.

These conjectures are supported by the empirical evidence shown in Fig 1.

Problem. Readers are invited to produce computer programs to determine the threshold of completeness of a given sequence and hence to evaluate a_{n-1} and obtain, in addition to the results given above (which are intended to provide test-data for debugging and optimising routines) corresponding tables of a_{n-1} for the sequences (i), (iv) . . . (viii). Conjectures similar to those above will be welcome together with theoretical proof of their validity! Submissions should be sent to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, Staffordshire WV4 5NF to arrive by 1 April 1988.

All submissions will be judged using subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution received by the closing date.

It would be appreciated if such submissions contained a brief summary of results obtained, in a form suitable for publication in *PCW*.

Please note that submissions can only be returned if a suitable sae is provided.

(ii) $f(n) = n^2$.					
n	50	100	150	200	250
s_n	2500	10000	22500	40000	62500
$T(S_n)$	17072	60928	129184	222208	339968
a_{n-1}	7.110	6.216	5.818	5.611	5.483
(iii) Prime Numbers					
n	100	500	1000	2000	
s_n	541	3571	7919	17389	
$T(S_n)$	1683	10779	23859	52247	
a_{n-1}	3.217	3.028	3.017	3.004	

when $a_{n-1} = T(S_n)/s_{n-1}$

Fig 1

Cyprian's Last Theorem, July 1987

Several contributors found the following results on sums of cubes:

$6^3 = 3^3 + 4^3 + 5^3$
$20^3 = 11^3 + \ldots + 14^3$
$40^3 = 3^3 + 4^3 + \ldots + 22^3$
$60^3 = 6^3 + 7^3 + \ldots + 30^3$
$70^3 = 15^3 + 18^3 + \ldots + 34^3$
$180^3 = 6^3 + 7^3 + \ldots + 69^3$.

Sums of squares yields only Pythagorean Triads discussed fully on other occasions. Searches for powers up to the seventieth failed to yield other solutions.

Dissection of the 6-cube into 3–, 4– and 5– cubes by John Cook in Australia claimed a result with 15 sections, 'one of which was rather irregular' (a letter to the editor from John explaining this would be appreciated!) Richard Tindell discovered an eight piece dissection and later a second in Lindgren's book entitled *Geometric Dissections* (page 24) and attributed to RE Wheeler in *Eureka* (1951).

There is no known dissection into fewer than eight pieces (I hope) but the number of distinct dissections into eight pieces is not clear to me.

However, this month's prize goes to Jonathan Hart of 72 Clifton Rd, Tunbridge Wells, Kent TN2 3AT, who, in addition to dissections of the cube (in fact into twelve pieces), carried out substantial investigation, both theoretical and empirical, on the remaining problem.

As a follow up to this work the Reverend D Cyprian Stockford asks whether integer solutions exist of: $p^2 + q^2 = r^2$ simultaneously with $p^3 + q^3 + r^3 = s^3$? Is this related to the historically famous: $p + q = n^2$ simultaneously with $p^2 + q^2 = m^4$? Before a search is started note that p = 1061652393520 and q = 4565486027761 is, in fact, the smallest solution.

Mike Mudge welcomes correspondence on any subject within the areas of numbers theory and other computational mathematics. Particularly welcome are suggestions, either general or particular, for future Numbers Count articles; all letters will be answered in due course.

Isolated readers can be put into contact with others sharing the same interests. However, greater efficiency regarding published problems should result from contacting the prizewinner.