LEISURE LINES

Brainteasers courtesy of JJ Clessa.

Quickie

Here's a message in code. Each letter corresponds to one letter from the original message. If I tell you that the letter 'A' in the coded version equals 'Z' in the original, can you decode it?

GFCKPQ D EJM EJFC DM MEL AJJ, D NDM MFCB MJJB JNN D NDGLG CLG NFC NLA.

Better still, just tell me the meaning of 'QCKP' — it's something you'll probably do when you've decoded the message.

Prize Puzzle

My thanks to Tim Higgins for the idea behind this puzzle.

The other day, the building society sent me notification of interest due on my account. Feeling affluent I rushed out and bought some booze to celebrate — spending an exact number of pounds in the process.

However, when I looked again at the letter, I realised I had mentally transposed the pounds and the pence, and the sum I was due was less than I'd thought.

In fact, I calculated that the amount I had thought I was getting, less the amount spent on booze, was an exact multiple of the amount that I actually received. And coincidentally, this multiple was the same as the pounds that I spent on booze.

If you've managed to understand all that, please tell me how much money I actually received, and how much I spent on booze.

Prize Puzzle, March 1988

First, a word about the April Quickie. I have already received many letters advising me how to use four 7s to generate a value of 26. But, alas, none of them fulfil the conditions of '. . . using standard mathematical symbols . . .'.

Most of you used 'log' which is not a symbol, or [], !! (double factorial), which are certainly not standard symbols. One reader, a maths teacher, even added a couple of extra digits (if that were permissible, Mr W, then $2 \times 77/7 + 7 - 3$ would have been simpler than your effort). The best — but not acceptable — was $7 + 7 + 7 + 7$ which is 26 in base II arithmetic.

So, although I'm still hoping that someone will come up with the goods, I'm beginning to doubt it.

Anyway, to the Prize Puzzle and the root of the Fibanacci sequences. The answer, by sheer number crunching, is 144 and 298. Of the 160-odd replies, 151 were correct. The lucky winner, drawn at random, was one of our regular entrants who, I believe, has won before — Mr Alan Northcolt of Winnersh, Berkshire.

NUMBERS COUNT

The fascinating topic of addition chains is explored by Mike Mudge.

Definition An 'Addition Chain' for a positive integer n is a finite sequence of positive integers:

$1 = a_0 < a_1 < a_2 < a_3 < \ldots < a_r = n$ where each member (other than $a_0 = 1$) is the sum of two earlier, but not necessarily distinct, members of the sequence.

Thus, two different addition chains for 14 are:

C_1: 1, 1 + 1 = 2, 2 + 2 = 4, 4 + 2 = 6, 6 + 2 = 8, 8 + 6 = 14

C_2: 1, 1 + 1 = 2, 2 + 2 = 4, 4 + 2 = 6, 4 + 4 = 8, 8 + 6 = 14

Each of these chains is said to have *length*, r = 5.

Definition The minimal length of an addition chain for n is denoted by L(n). A 'Brauer chain' is one in which a shortest chain exists where each member uses the previous member as a summand.

Note that C_2 above is not a Brauer chain because 4 + 4 = 8 does *not* use the previous term — that is, the 6 — but it is a minimal chain.

Any number n which has a Brauer chain is called a 'Brauer number'.

Definition An addition chain for which there is a subset H of the members, such that each member of the chain uses the largest element of H which is less than the member, is called a 'Hansen chain'.

Note that C_2 above is a Hansen chain with H = (1,2,4,8).

Donald Knuth, in *The Art of Computer Programming Vol 2* (Addison-Wesley 1969, pp398-422) gives the following addition chain for 12509:

1, 2, 4, 8, 16, 17, 32, 64, 128, 256, 512, 1024, 1041, 2082, 4164, 8328, 8345, 12509.

This is *not* a Brauer chain since 32 does not use 17. However, it *is* a Hansen chain with H = (1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 1041, 2082, 4164, 8328, 8345).

No Brauer chain of length 17 or less exists for 12509.

A conjecture of Arnold Scholz (1937)

The minimal length of an addition chain for $2^n - 1$ differs from the minimal length of an addition chain for n by less than n.

$L(2^n - 1) \leq n - 1 + L(n)$.

A question of Richard Guy (1983)

Are there any numbers n which do not have Hansen chains? That is, are there any Non-Hansen numbers?

Note the Scholz Conjecture has been proved for $n = 2^a$, $2^a + 2^b$, $2^a + 2^b + 2^c$ and $2^a + 2^b + 2^c + 2^c$ by Utz, Gioia *et al* (1953) and demonstrated for $1 \leq n \leq 18$ and n = 20,24 and 32 by Knuth & Thurber (1973/76).

Problems

(i) Construct a computer program to obtain all possible addition chains for a given n. The complete output should only be generated for certain small values of n for test purposes!

(ii) Modify the above program to list any Brauer and/or Hansen chains produced. In the latter case, the appropriate subset H should be output.

(iii) Establish the value of L(n) as a function on n and hence verify the Scholz conjecture, albeit for a small range of n.

(iv) Comment upon the empirical evidence for the existence of Non-Hansen numbers.

Readers are invited to send their attempts at some, or all, of the above problems to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, South Staffordshire WV4 5NF, tel: (0902) 892141, to arrive by 1 September 1988. It would be appreciated if such submissions contained a brief description of the program and a summary of the results obtained in a form suitable for publication in *PCW*. These submissions will be judged using subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution received by the closing date.

Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Review: December 1987

Attempts to investigate the sequence directly are clearly doomed, x_{17} having 2661 digits. However, H Ibstedt determined that x_{42} with 89288343500 digits would, if printed 80 digits per line and 60 lines to a page, require more than nine million sheets of paper and weigh approximately 35000kg!

But the most comprehensive study was that of H Ibstedt of 4 Rue Gramme, Paris 75015, whose results include the location of the first non-integer term for all powers up to the eleventh and initial values x_1 from 2 up to 11. A very worthy prize-winner.

The longest integer sequence of 600 terms occurs for cubes and $x_1 = 11$, while the shortest of 7 occurs several times in the above study.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and other computational mathematics. Particularly welcome are suggestions, either general of specific, for future Numbers Count articles; all letters will be answered in due course.

Isolated readers can be put in contact with others sharing the same interests. However, greater efficiency regarding published problems should result from contacting the prizewinner.