Prime time

Mike Mudge mulls over prime divisors of binomial coefficients.

What are binomial coefficients?
Binomial coefficients are an array of positive integers denoted by ${}_nC_r$ or $\binom{n}{r}$ defined by $\frac{n!}{(n-r)!\,r!}$: and either considered as counting the number of combinations or selections of n unlike things taken r at a time, or the coefficient of $x^r y^{n-r}$ in the expansion in ascending powers of x of $(x + y)^n$. $0 \leq r \leq n$.

The calculation of ${}_nC_r$ may be carried out using the above definition where, of course, N! denotes the product $1\times2\times3\times4\times\ldots\times N$ and is read as factorial – N: or using a convenient button on a pocket calculator: or using Pascal's Triangle; thus,

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      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1  etc.
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$${}_{n-1}C_r + {}_{n-1}C_{r-1} = {}_nC_r$$

each term (other than the bounding 1's) in a given row being the sum of the two terms immediately above. The row count corresponds to n and the diagonal count to r, thus ${}_4C_2 = 6$.

The calculation of, say, ${}_{44}C_6$ using Pascal's Triangle would require the use of 44 rows (an interesting formatting exercise in Basic!): however,

$${}_{44}C_6 = \frac{44!}{38!\ \ 6!}$$
$$= \frac{44\times43\times42\times41\times40\times39}{6\times5\times4\times3\times2\times1}$$
$$= 44\times43\times7\times41\times13$$
$$= 7059052$$

Problem 1 The construction of an efficient algorithm to generate the expansion into prime factors of ${}_nC_r$. Interested readers may consult Pierre Goetgheluck, *American Mathematical Monthly*, volume 94, 1987, pp360–365.

Problem 2 The graphical representation of the distinct prime factors, p, of ${}_nC_r$. *Hint* Using a different set of p, r – axes for *each* value of n, the point (p,r) is plotted if, and only if, p divides ${}_nC_r$.

How does the general shape of this graph evolve as n increases?

Problem 3 Tabulation of w(n,r) the total number of distinct prime factors of ${}_nC_r$.

Problem 3* Erdös has found that w(2n,n) approaches E(n) = n log(4)/log(n) as n becomes very large: obtain empirical evidence for this.

Test data	n	w(2n,n)	E(n)	%Error
	500	116	112	3.45
	1000	208	201	3.37

Problem 4 Erdös conjectured that for all $n > 4$ it is true that ${}_{2n}C_n$ is never square-free: obtain empirical evidence for this. *Note* It is known to be true if (i) $4 < n < 2^{42205184}$ or (ii) $n \neq 2^a$ but verification is an exercise in efficient programming!!

Readers are invited to send their attempts at some, or all, of these problems to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, South Staffordshire WV4 5NF, tel: (0902) 892141, to arrive by 1 March 1989.

It would be appreciated if such submissions contained a brief description of the hardware used, details of programs, run times and a summary of results obtained, together with suggestions for further investigation, all in a form suitable for publication in *PCW*.

These submissions will be judged using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution received by the closing date.

Please note that submissions can only be returned if a suitable stamped, addressed envelope is provided.

Review: July 1988

This review will be concerned only with a report of the results obtained by the very worthy prize winner, Gordon Mills of Lyndon House, Catbrook, Chipping Campden, Glos GL55 6DG.

This major computing effort of 'well over 100 hours' using Basic 2 on an Amstrad PC1640 with integer limitation of 2^{31} yielded many interesting results, including those shown in the box below.

Empirical analysis included plotting n against the logarithm of both the minimum sum and minimum product; this casts doubt on the possible results quoted in the box.

Can any readers help Gordon to further extend these results, and possibly publish a paper on this subject?

Minimum sums of n-tuples from 7 to 11 as:

n	sum	product
7	511	1965600
8	1022	15724800
9	1287	34927200
10	2574	279417600
11	5148	2235340800

Possibles include:

n	12	16	20	28
sum	9282	1137708	271085958	315599627733390

Stop Press
AP Bermingham of 25 Murray Mews, London NW1 9RH, has addressed the problem of generating prime numbers through programming in Hypertalk on a Mac II. The two techniques used are: (i) The Sieve of Eratosthenes; and (ii) The Wilson Congruence ($(p-1)! \equiv p - 1$ (modulo p)).

How do other readers generate prime numbers?

Mike Mudge welcomes correspondence on any subject within the areas of number theory and other computational mathematics. Particularly welcome are suggestions, either general or specific, for future Numbers Count articles. All letters will be answered in due course.

Brainteasers courtesy of JJ Clessa.

Quickie
What is the smallest number that uses the vowels A E I O U, and Y, once only, when it is written down? (For example, Twenty Four uses E Y O and U.)

Prize Puzzle
A few months ago we published a quickie in which four 7's were used to generate the successive numbers 1-20. This month's prize puzzle is a variant of the theme.

Using as few 7's as possible, together with the mathematical symbols shown below, generate exactly the value 13579.02468.

Permitted symbols are + – / * ! √. ( ). No others, and only digit 7 may be used. For example, if we'd asked you to generate 100.01 the answer could have been:

$$\frac{77}{.77} + \frac{.77}{77}$$

Answers on postcards only, please, to arrive not later than 31 January 1989. Please state clearly how many 7's were used.

Prize Puzzle, October 1988
This problem didn't present too much difficulty when attacked by computer trial and error. Just less than 100 entries were received, with the winning card coming from Mr O'Connell of Shanklin, Isle of Wight. Congratulations, Mr O'Connell, your prize is on its way.

There are six possible solutions, each of which gives 16 occurences. We accepted any, although most readers sent them all in. They are:

1702
2017
2503
2764
3025
3367

To all the runners-up, keep trying — you could be next.