Untouchable Numbers

Mike Mudge investigates the Aliquot Parts of positive integers, a topic which prompts questions first asked in 1657 and which requires clever use of a computer for verification.

Definition (i) The Aliquot Parts (or factors) of a positive integer are defined to be those positive integers, including unity, which are less than the integer and divide it exactly — that is, without remainder.
Definition (ii) Given a positive integer, n, the sum of its Aliquot Parts shall be written S(n).

Paul Erdös has proved (1979) that there are infinitely many m such that the equation $S(x)=m$ has no solution. Jack Alanen (1972) called such m Untouchable Numbers.

There are known to be 89 untouchable numbers less than one thousand, the sequence begins: 2,5,52,88, . . .

Problem 1 Write a computer program to input a positive integer, n, and to output S(n). Hence determine all the untouchable numbers less than one thousand.

Unsolved problems relating to untouchable numbers include:
a) Is 5 the only odd untouchable number?
b) Are there arbitrarily long sequences of consecutive even numbers which are untouchable?
c) How large can the gap between consecutive untouchable numbers be?

Note In much of the following work, n, will be expressed as the product of its prime factors rather than as an explicit integer, thus 888 will be considered as $2^3.3.37$ and 1444 as $2^2.19^2$.

Readers are therefore encouraged to input, store and output integers in this form and further to address the problem of computing S(n) if $n=p_1^{e1}.p_2^{e2}.p_3^{e3} \ldots p_r^{er}$, where the $p_1 \ldots$ to p_r are the distinct prime factors of n, and $e_1 \ldots$ to e_r are their multiplicities. It is not desirable to construct the integer explicitly but rather to choose the possible combinations from the known prime factors.

On 3 January 1657 Pierre Fermat asked for solutions of:
A) $n^3+S(n^3)=m^2$; also of
B) $n^2+S(n^2)=m^3$

On 17 March 1657 John Wallis asked for solutions of:
C) $m^2+S(m^2)=n^2+S(n^2)$

Note With reference to A that $7^3+S(7^3)=7^3+(1+7+7^2)=400 =20^2$; and with reference to C that $4^2+S(4^2)=4^2+(1+2+4+8) =31$ while $5^2+S(5^2)=5^2+(1+5) =31$ also.

A larger and more typical solution of C) due to Frenicle (1658) is given by $n=2.163$, $n^2+S(n^2)=2^2.163^2+1+2+163 +2^2\ 163^2+2.163+2^2 .163+2.163^2=187131$, while $m=11.37$, $m^2+S(m^2)=11^2.37^2 +1+11+37+11^2+37^2+11.37+ 11^2.37+11.37=187131$ also.

Many solutions to A, B and C are known and it is *not* anticipated that readers of *PCW* will add to this body of knowledge.

Problem 2 Write a computer program to input a positive integer, n, in the form of the product of its prime factors and to evaluate, and output, $n^2+S(n^2)$. Hence verify the correctness of the following solutions to C.
n: 3.11.19; 2.5.151; 2^3.3.37; 29.67; 2^3.7.29.67.
m: 7.107; 3^3.67; 2.19.29; 2.3.5.37; 3.5.11.19.37.

Problem 3 Use the program developed in solving problem 2 above to construct m-values given that the following n provide solutions to B.
n: 7.11.29.163.191.439; 43098; 2^2.5.7.11.37.67.163.191 .263.439.499.

Problem 4 Discover or verify the solution to A given below.
n: 2^5.5.7.31.73.241.243.467;
m: $2^{12}.3^2.5^3.11.13^2$.17.37. 41.113.193.257;

Attempts at some, or all, of the above problems may be sent to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, South Staffordshire WV4 5NF, tel: (0902) 892141, to arrive by 1 July 1989. Any submissions received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date.

It would be appreciated if such submissions contained a brief description of the hardware used, details of programs, run times and a summary of results obtained; together with suggestions for further work, all in a form suitable for publication in *PCW*.

Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Review, November 1988
The iterative procedure, suggested by Paul Cleary and forming the basis of this article, did not find widespread appeal among *PCW* readers. I wonder why? However, the work of Gareth Suggett of 31 Harrow Road, Worthing, Sussex BN11 4RB deserves the prize award. Gareth claims to have established that sums of squares of factors will always diverge and doubts whether neglecting multiplicities would be sufficient to produce a convergent sequence.

He also suggests a related problem, iterating: $X_n=\text{Int.}(F(X_{n-1}))$ where $F(X)=a_1p_1^y+ a_2p_2^y+ \ldots +a_np_n^y$ in which $X=p_1^{a1}p_2^{a2} \ldots p_n^{an}$ and y is a real number between 1 and 2.

A recent appeal from Ron J Cook of 113 Critchill Road, Frome, Somerset BA11 4HW, tel: (0373) 64351, has been seen:

'Multiple Precision (or extended) Arithmetic. I understand only the basic principles of this technique . . . I should like to know the principles and structure of these programs . . . a reference to a book or paper, an expert, any lead, I should be extremely grateful . . .' Surely there are many *PCW* readers who can help Ron with this problem.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and other computational mathematics. Particularly welcome are suggestions, either general or specific, for future Numbers Count articles. All letters will be answered in due course.

LEISURE LINES

Brainteasers courtesy of JJ Clessa.

Quickie
No answers, no prizes.
When the day after tomorrow is yesterday, today will be as far from Wednesday as today was from Wednesday when the day before yesterday was tomorrow. What day is it now?

Prize Puzzle
There's a certain number X which is the product of 4 different prime numbers (units excluded).
The square of X contains 9 digits, the first three of which are the same as the last three, and the middle three equal the sum of the first and last three (that is, twice the first three). Got it?
What's the largest number that X could be?
Answers on postcards only (or backs of envelopes) to: Leisure Lines Prize Puzzle May 1989, *PCW* Editorial, VNU House, 32-34 Broadwick Street, London W1A 2HG, to arrive not later than 31 May 1989.
Good luck!

Prize Puzzle, February
Only 63 entries for this month's problem, and of these only 46 had the correct solution.
The answer was 11579 inches (or, as some of you insisted on putting it, 321 yards 1 foot 11 inches).
The winning solution came from Anthony Isaacs of London. Congratulations, Mr Isaacs, your prize is on its way. To all the also-rans — keep puzzling, it could be your turn next.