Permutations leading to some famous integer sequences, or 'Have you done any anagrams recently?', from Mike Mudge.

Definition A permutation of n distinct objects is defined to be any particular arrangement of them. (Note: ordering is of fundamental importance in an arrangement, which is why a combination lock should strictly speaking be called a permutation lock, and why any anagram can, in principle, be solved by constructing a complete listing of all possible permutations of the letters involved.)

Since in any particular arrangement (permutation) of n distinct objects there is a choice of 1 from n for the first position, 1 from n-1 for the next position, 1 from n-2 for the next position and so on, with finally 1 from 1 from the n^{th} or last position, it follows that a total of
$n \times (n-1) \times (n-2) \times \ldots \times 1 = n!$
(factorial n) permutations are possible.

Notation Suppose that the n distinct objects are represented (named or labelled) by the positive integers 1,2,3,4....n. A particular permutation may then either be specified by a table in natural order or by a sequence of cycles. Thus the ordering 3,5,4,1,2,6,7 of the first seven positive integers is represented either by the natural order table:
1 2 3 4 5 6 7
3 5 4 1 2 6 7
or by the sequence of cycles
(1 3 4) (2 5) (6) (7),
both of which mean that when starting from the natural ordering replace 1 by 3, 3 by 4, 4 by 1, 2 by 5, 5 by 2, 6 by 6 and 7 by 7.

Stirling Numbers of the First Kind, S(n,k) are defined to be the number of permutations of n distinct objects containing exactly k cycles. For example, if n = 3 the possible 3! = 6 permutations are:
(123);(132);
(1)(23);(2)(13);(3)(12); and
(1)(2)(3).
Hence S(3,1) = 2, S(3,2) = 3 and S(3,3) = 1.

Stirling Numbers of the First Kind

n/k	1	2	3	4	5	6	Row Sum =n!
1	1						1! = 1
2	1	1					2! = 2
3	2	3	1				3! = 6
4	6	11	6	1			4! = 24
5	24	50	35	10	1		5! = 120
6	120	274	225	85	15	1	6! = 720

Subfactorials or Recontres Numbers, R_n are defined to be the number of permutations of n distinct objects in which every object is moved from its natural (or original) position. Such a permutation is technically called a derangement.
Thus $R_4 = 9$ is displayed in tabular form as follows:
2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, and 4321.

n	2	3	4	5	6.....
R_n	1	2	9	44	265.....

Euler Numbers, E_n are defined to be the number of permutations of n distinct objects whose associated integers first rise and then alternately fall and rise. (This becomes irrelevant if the objects are considered to be the positive integers themselves — that is, the latter are not merely labels.)
Thus $E_4 = 5$ is displayed in tabular form as follows:
1324, 1423, 2314, 2413 and 3412: each permutation shown exhibits the required rise, fall, rise . . . pattern of behaviour.

n	1	2	3	4	5	6
E_n	1	1	2	5	16	61

The odd Euler numbers are called tangent numbers, $T_n = E_{2n-1}$, because
$\tan x = 1\,x/1! + 2\,x^3/3! + 16\,x^5/5! \ldots$
The even Euler numbers (frequently called simply the Euler numbers) should, by analogy, be called secant numbers since
$\sec x = 1 + 1\,x^2/2! + 5\,x^4/4! + 61\,x^6/6! \ldots$

Problem (1) Write a computer program which, by constructing and counting the appropriate permutations, obtains S(n,k), R_n & E_n.

Problem (2) By examining the algorithm/ output of (1) above and obtaining an algebraic generating function or otherwise, obtain an optimal algorithm for generating and factorising S(n,k), R_n & E_n.

Attempts at the problem and/or thoughts on the 'Stop Press' question may be sent to Mike Mudge, 'Square Acre', Stourbridge Road, Penn, South Staffordshire WV4 5NF, tel (0902) 892141, to arrive by 1 February 1990. Any communications received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date.

Review, June 1989 — Hofstadter Sequences
Among the responses to this problem, that of Frank Webster's BBC Basic on an Acorn Electron deserves special mention. Values of a_n up to n = 30000 in 16mins. Conclusion: infinite number of randomly distributed absent values, 5000 terms of b_n in 75hrs. Suggestion of count of missing numbers asymptotic to a straight line representing an increase of 21 missing numbers per 1000 increase in search. Those c_n less than 20000 found in 22hrs; suggestion that number of primes less than N is a major factor in determining the number of missing numbers.

A Parry using MacPascal on a MacPlus then BBC Basic on an Archimedes A310 found the first 145,000 terms (a_n) and 100000^+ terms (b_n).

The very worthy prizewinner is Anthony Quas of Woodcroft, Weston-in-Gordano, Bristol BS20 8PZ, using 68000 assembler and APL 68000 on a 2Mbyte Atari ST running Mirage. Up to ^{c}200000 in 10 mins. Proof that a_n/n tends to ½ and that $a_n/n - ½$ could be regarded as a normally distributed random variable . . . Up to ^{c}6000000 in 20 mins. Further that the fraction of the integers occurring tends to about 0.54. A very detailed piece of work, full details from Mike Mudge or Anthony Quas.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and other computational mathematics. Particularly welcome are suggestions, either general or specific, for future Numbers Count articles. All letters will be answered in due course.

LEISURE LINES

Brainteasers courtesy of JJ Clessa.

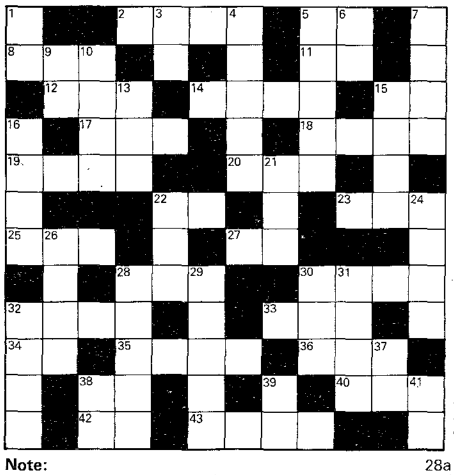
Winner, September 1989
We had the problem this time — not you. We omitted to put sufficient constraints on the puzzle and it turned out that more than a thousand solutions satisfied the rules as stated — 1609 to be exact, our most reliable entrant tells us. (Thanks, SNH.)

Probably as a result of this confusion, the total entry was quite a bit down from usual. The winning card came from Mr Bruce Halsey of Norfolk who sent us several of the solutions possible — including the right one, which was £4.95. Congratulations Mr Halsey, your prize is on its way.

Prize Puzzle, December 1989
This month's problem should be attempted when you are full of turkey, booze and Xmas festive spirit, and need something to keep your brain alive. It's a number crossword. You may use your micro to help you, but you'll probably find that it's not really necessary.

Stick the completed grid to a postcard or to the back of a sealed envelope — not in a letter please — and send it to arrive not later than New Year's Eve 1989, to: December Prize Puzzle, *PCW* Editorial, VNU House, 32–34 Broadwick Street, London W1A 2HG.

Clues Across
2: 5a squared
5: Half of 13d
8: 1d squared
11: 3d–1d
12: One third of 14a
14: 19a with digits rearranged
15: 11a–61d
17: Same as 12a
18: 10 times 25a
19: 2a plus 20a
20: 27a squared
22: 1d–6d
23: 5 times 15a
25: 28a+39d
27: Same as 15a
28: 21d+40a
30: 7d–15d
32: 30a–37d–15d
33: 30d+100
34: 39d+41d
35: 2a+14a+1d+16d
36: Last three digits of 31d
38: One third of 11a
40: One third of 25a
42: 38a times 8
43: Square of 42a



Clues Down
1: One third of 9d
3: Reverse of 1d
4: 4 times 2a
5: 40a squared
6: 9d–3d
7: 10 times 12a
9: Square root of 10d
10: Twice 19a
13: Square root of 4d
15: 10 times 40a
16: 1d times 5a
21: 40 less than 20a
22: 40 more than 13d + 15a
24: 18a–16d
26: 16d+7d
28: 1d times 2a
29: Square of 22d
30: 37d squared
31: 7d–9d
32: 39d squared
37: 27a plus 2
38: One quarter of 34a
39: 34a–41d
41: 5a–11a

Note:
All answers are whole numbers.
28a = the answer to 28 across.
16d = the answer to 16 down.