An investigation into the 'not-often' monotonic† sequence of numbers, conducted by Mike Mudge.

† **Definition** *A sequence of numbers $\{t_r\}$, $r = 0,1,2,3...$ is said to be monotonic increasing if and only if $t_{r+1} > t_r$ for all values of r.*

The following area of investigation has been suggested by a former colleague of mine, Robert Vein of the Department of Computer Science and Applied Mathematics at the University of Aston in Birmingham.

The mathematical background to the sequences being investigated, involving the concept of an n[th] order Hessenberg determinant, is contained, in essence, in Appendix I. However, an understanding of this work is not needed in order to carry out the computational investigation, which provides an opportunity to use general integer length arithmetic coupled with skilful programming and ingenious choice of input data.

Given a sequence $\{g_r\}$, $r = 0,1,2,3...$ a related sequence, $\{H_r\}$, is defined implicitly as follows:

$g_0H_0 = 1$
$g_1H_0 - g_0H_1 = 0$
$g_2H_0 - g_1H_1 + g_0H_2 = 0$
$g_3H_0 - g_2H_1 + g_1H_2 - g_0H_3 = 0$

....

For one possible justification of these relationships see Appendix I.

Explicitly the above equations may be solved for the H – sequence to yield: $H_0 = 1/g_0$ &

$$H_r = (-1)^{r-1} \sum_{s=0}^{r-1} (-1)^s g_{r-s} H_s / g_0$$

Now, Robert initially considered the case where $\{g_r\}$, $r>0$, is the sequence of Prime Numbers and where for convenience g_0 is defined to be 1. The results in Fig 1 were obtained for this case where $\{g_r\}$ is denoted by $\{p_r\}$.

Note that $\{H^*_r\}$ is apparently monotonic increasing for $r>2$.

r	0	1	2	3	4	5	10	15	20	25	30	35
p_r	1	2	3	5	7	11	29	47	71	79	113	149
H_r	1	2	1	1	2	3	26	193	1149	7352	48969	320973

Fig 1

* *Based upon experimental evidence up to r = 50.*

Problem 1 Reproduce the above results, extend them as far as possible and attempt to prove the above observation regarding monotonicity.

Problem 2 Assuming that the $\{H_r\}$ generated from the sequence of primes, that is when $g_r = p_r$, is monotonic increasing from H_3 onwards, does there exist a 'smaller' monotonic sequence $\{q_n\}$; in the sense that $n < q_n \leqslant p_n$, with $q_n < p_n$ for at least one value of n, and such that the associated sequence $\{H_n\}$ is monotonic from one particular value of n onwards?

Now recall the Fibonacci Sequence, $\{F_r\}$, defined by $F_0 = 1$, $F_1 = 1$ while $F_r = F_{r-1} + F_{r-2}$ for all $r \geqslant 2$ (see 'Numbers Count', *PCW* May 1983).

Problem 3 Investigate the sequence $\{H_r\}$, generated by the above relations in the case where $g_r = F_r$ for all r.

Problem 4 An open-ended investigation. Choose *any* sequence 'that come to mind' (always provided that they are uniquely defined) in the hope of finding another apparently (or even probably) monotonic increasing sequence $\{H_r\}$; or, failing that, some other interesting output.

For example, investigate the case where $g_r = a^r$ for $a = 2,3,4...$ followed by that where $g_r = b + c \times r$ for small positive integers b & c.

Attempts at some or all of these problems, together with any other observations on this area of investigation, may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel 0994 231121, to arrive by 1 August 1991.

Any communications received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date.

It would be appreciated if such submissions contained a brief description of the hardware used, details of programs, run times and a summary of the results obtained; together with suggestions for further work in this area, all in a form suitable for publication in *PCW*. Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Appendix I: a sequence of Hessenberg Determinants

Let $\{g_n\}$, $n \geqslant 0$, denote a given sequence of positive integers. Define the associated sequence, $\{H_n\}$, $n \geqslant 0$, by means of the identity: $(g_0 + g_1x + g_2x^2 + g_3x^3...)(H_0 - H_1x + H_2x^2 - H_3x^3...) = 1$.

Algebraic expansion of the product on the left hand side, followed by comparison of the coefficients of each power of x, yields a sequence of relationships between the H_n:

$g_0H_0 = 1$
$g_1H_0 - g_0H_1 = 0$
$g_2H_0 - g_1H_1 + g_0H_2 = 0$

....

The H_n so defined may be represented explicitly as a sequence of determinants, known as Hessenberg Determinants (Fig 2).

$$H_n = \begin{vmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & \cdots \\ g_0 & g_1 & g_2 & g_3 & g_4 & \cdots \\ 0 & g_0 & g_1 & g_2 & g_3 & \cdots \\ 0 & 0 & g_0 & g_1 & g_2 & \cdots \\ 0 & 0 & 0 & g_0 & g_1 & \cdots \\ \cdots & & & & & \cdots \end{vmatrix}_n$$

The compact notation being: $H_n = |h_{ij}|_n$ where $h_{ij} = g_{j-i+1}$ if $i \leqslant j + 1$ else $h_{ij} = 0$

Fig 2

LEISURE LINES

Brainteasers courtesy of JJ Clessa.

Very, Very Quickie

No answers, no prizes. Just find a single word anagram of MONDAY.

Prize Puzzle, June 1991

This month's problem should *definitely* get the micros whirring. Take any positive integer, reverse it (using leading zeros if necessary) and add it to the original number. Then, repeat the operation with the result, and so on. Eventually, you will always arrive at an answer which is palindromic — well, nearly always.

To illustrate:

Take the integer 28: reverse it — 82: add the two — 110

110: reverse it — 011: add the two — 121

This is palindromic after only two operations.

Try another:

273: reverse it — 372: add — 645

645: reverse it — 546: add — 1191

1191: reverse — 1911: add — 3102

3102: reverse — 2013: add — 5115 which is palindromic.

Now, we said that it works nearly always. But there is a number which does not conform even after thousands of steps. What is it? And, so that you'll be able to send in the result before the 30 June deadline, we'll tell you that the number is less than 1000.

Just write the number on a postcard or the back of a sealed envelope and send it to June Prize Puzzle, *PCW* Editorial, VNU House, 32-34 Broadwick Street, London W1A 2HG.

Winner, March 1991

The required digit arrangements are: 039247185 and 092346185 and hence the digits which only appear once are 6 and 7.

Winner, by random selection, was Mr JC McCarthy of Mansfield, Notts, who receives our congratulations immediately and a fine prize very shortly. ■