The rise and fall of Hailstone Numbers

This title is that of Chapter 3 in the book entitled *Think of a Number* by Malcolm E Lines, Adam Hilger 1990, £7.50; an interesting bedtime read for all those interested in 'Numbers Count'.

The Collatz problem, first visited in *PCW* August 1984, reads thus: Think of a number; if it is odd, triple it and add one; if it is even, halve it and repeat this process until the initial number is recovered... if ever? An interesting algebraic formalism of this iterative scheme is due to David Fisher of Cardiff:

$$x_{n+1} + \tfrac{1}{2}\left(x_n . 3^{(1 - \cos \pi x_n)/2} + (1 - \cos\pi x_n)/2 \right).$$

Now, starting with x_0 as 1, 2 or 3 easily leads to the loop 1,4,2,1; and even starting with 7 yields the same loop quickly although a maximum value of 25 is reached in so doing. However, with $x_0 = 27$ the high of 9232 falls back to $x_{111} = 1$.

Problem A

Investigate the behaviour of 'Hailstone numbers' with starting values 1,2,3... (In the University of Tokyo researched up to $x_0 = 10^{12}$). Graph x_n against n for a range of values of x_0 and hence interpret the name 'Hailstone Numbers'. Alternative names (printable in *PCW*!) for these erratically behaving numbers are invited.

Problem B

By graphical analysis, or otherwise, construct the 'best possible' (from your data) linear relationship between the sequence length and x_0, paying particular attention to the behaviour for large x_0.

Problem C

Replace the coefficients 3 & 1 in 'if x_n odd then x_{n+1} becomes 3 x x_n + 1' by other (small) odd numbers and search for those cases generating a multiplicity of loops. For example, when the above coefficients are 3 &7 respectively the value $x_0 = 1$ leads to 5,22,11,40,20,10,5, while $x_0 = 7$ leads to 7,28,14,7.

Responses to some or all of the above problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121 to arrive by 1 April 1992.

Any communications received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date.

It would be appreciated if such submissions contained a brief description of the hardware used, details of programs, run times and a summary of the results obtained; together with suggestions for further work in this area, all in a form suitable for publication in *PCW*. Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Review, June 1991: a 'Not-Often' Monotonic Sequence due to Robert Vein

This apparently obscure area of investigation produced numerous detailed responses, including those from:

- N Backhouse, who investigated $g(x) = \sum H_r x^r = 1/f(x)$ where $f(x) = \sum^N p_r x^r$ (p_r being the r^{th} prime and N = 100) and conjectured that $H_{r+1}/H_r \sim C$, C is approximately equal to 3/2.
- G Suggett, who found a number of relevant results, including the case $g(x) = \sum p(n)x^n$, where p(n) is the number of partitions of n, in *Hardy & Wright*.
- N Hodges discovered a sequence {g} 'which is virtually as small as possible and where {H} grows very slowly', but was unable to find {g} & {H} which both grew linearly at worst.
- B Stewart, after a limited theoretical analysis, began computing in Mathematica v1 on a Viglen Genie 386 with 387 co-processor and 4 Mb of RAM. In about 25 minutes he found the first 1000 coefficients related to $1 + \sum p_n x^n$... some very large numbers, and observed: 'I would take modest bets, though, that the matrix formulation you give is a complete distraction'. Any wagers?
- R Bond investigated a range of $\{g_i\}$ uncluding Primes, Lucas Numbers, Polynomials of degree 1,2 & 3, Exponential and Factorial functions and a first order difference equation.

However this month's prizewinner is David Broughton of Freshwater, Isle of Wight. Using C and the Lattice C compiler version 2.01 on an IBM PC, David exhibited evidence to support the remarkable result that if G() has more than one root in the range[0,1] then H_{r-1}/H_r tends to the smallest root.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.