The problem with plates

Squaring the Rectangle! A problem of Geometrical Dissection, presented by Mike Mudge.

The problem this month is to divide a given rectangular plate into a set of square plates all of different sizes. The requirement that the sides of the resulting squares are to be rational multiples of those of the given rectangle means that there is no loss of generality in working throughout with integers.

Interest in this classical dissection problem, dating from 1925, has been rekindled by the recent publication *Unsolved Problems in Geometry* by HT Croft, KJ Falconer and RK Guy, Springer-Verlag 1991, where it is stated that 'The advent of computers allowed extensive searches to be made'.

The ORDER of a squared rectangle is defined to be the number of squares into which it is divided. The rectangle is PERFECT if no two squares are equal and IMPERFECT if this is not the case; further, it is a COMPOUND rectangle if it contains a squared sub-rectangle and SIMPLE if it does not.

The first perfect squared rectangle, 65 x 47 and of order 10 was published in 1925 by Z Moron who is also responsible for the 32 x 33 squared rectangle of order 9, now known to be the smallest possible order.

The notation to be adopted (avoiding the need for detailed diagrams) involves listing the coordinates of the vertices of each square in the dissection relative to the bottom left-hand corner of the given rectangle. These coordinates will be contracted along left to right rows thus: (0,0),(9,0),(17,0),(32,0) will be written (0,9,17,32;0). Plotting will reveal that (0,9,17,32;0), (9,10,17;8), (0,9,10;9), (10,14,17,32;15), (0,10,14;19) and (0,14,32;33) defines the perfect dissection of the 32 x 33 rectangle, referred to above, into squares of sides 1,4,7,8,9,10,14,15 & 18 respectively.

The search for a squared square (SS) of lowest order led RP Sprague in 1939 to an SS of order 55, TH Willcocks in 1951 to an SS of order 24, and ultimately, following upon an extensive computer search, to AJW Duijvestijn attaining the minimum possible order of 21 in 1978.

Questions posed by Z Moron:

M(i) In a squared rectangle are there always at least two squares which are surrounded by (four) larger ones?

M(ii) In a squared rectangle whose sides are relativley prime, is there always a square whose side is a perfect square?

M(iii) Does there exist a rectangle which can be squared in two distinct ways? That is, with no particular square occurring in both of the configurations.

Mrs Perkins' Quilt (HE Dudeney, Problem 173, *Amusements in Mathematics*, Thomas Nelson and Sons, 1917). This problem considers IMPERFECT squarings of a square. Let f(n) be the smallest number (greater than one) of integer sided squares into which a given integer sided square can be cut. The dissections are to be prime (i.e. the greatest common factor of the square sides is one, so that the dissection is not a scalar multiple of another solution). NONPRIMITIVE solutions which contain subdivisions that are scalar multiples of smaller solutions are allowed.

It is known that $\log_2(n) \le f(n) \le 6\log_2(n)$, the lower bound is due to JH Conway, Proc.Cam.Phil.Soc. v60 1964 pp363-368 and the upper bound to GB Trustrum, Proc.Cam.Phil.Soc. v61 1965 pp7-11.

The actual values of f(n) are known, with some increasing lack of confidence, for n≤100:

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n      1  2  3  4  5....
 30-39,41.....88-100
f(n) 1  4  6  7
 8.... 15...... 19
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Two rectangles are INCOMPARABLE if neither will fit inside the other with sides parallel, i.e. $a_1 x a_2$ and $b_1 x b_2$ where $a_1 < b_1 \le b_2 < a_2$.

Now dissect a rectangle into at least two incomparable rectangles. No rectangle can be so dissected with 6 or fewer resulting rectangles.

Questions posed and answered by ACC Yao, EM Reingold and BS Sands, *J Recreational Mathematic*, v8 1975/76 pp112-119:

YRS(i) Show that the smallest rectangle that can be imcomparably dissected is 22 x 13. (9 x 6, 2 x 16, 7 x 7, 10 x 5, 11 x 4, 3 x 13, 1 x 18)

YRS(ii) Show that the smallest square that can be incomparably dissected into 7 rectangles is 34 x 34.

YRS(iii) Show that a 27 x 27 square can be dissected incomparably into 8 rectangles. (1 x 27, 3 x 23, 7 x 17, 18 x 6, 21 x 4, 11 x 9, 16 x 8, 5 x 19). Is this the smallest square that can be dissected into incomparable rectangles?

Attempts at coding and reproducing 'Squared Rectangles' from the literature, at Problems M(i/ii/iii), at reproducing the table of f(n) values and finally at Problems YRS (i/ii/iii) may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 June 1992.

Any communications received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date. It would be greatly appreciated if such submissions contained a brief description of the hardware used, program listings, run times and a summary of the results obtained; together with suggestions for further work in this area, all in a form suitable for publication in *PCW*. Please note that submissions can only be returned if a suitable stamped addressed envelope is provided.

Review, November 1991

The subject area of RIT's, ARIT's and GRITs due to BG Anderson began with David Salkeld examining the crystalline structure of metals at Hong Kong University. This generated a very popular research topic with numerous responses of a high standard.

David Broughton, and others, demonstrated in lengthy submissions that the article examined 'only a tiny subset of GRITs'. Nigel Hodges gave a detailed theoretical exposition involving Pellian equations together with much computation. Pal Gronas, from Norway, derived necessary and sufficient conditions for an ARIT and proved that there are infinitely many GRITs of Type A. E Knighting dealt algebraically with Problem E only, generating an infinity of solutions.

After much soul-searching, and considering clarity and simplicity of approach, this month's prizewinner is Mike Bennett of Wokingham, Berks. His submission contained a balanced combination of Pellian equations and computation involving Cambridge Lisp on an ARM second processor connected to a BBC Master computer.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.