A midsummer miscellany

Five Unsolved Number problems, presented by Mike Mudge.

This month readers are invited to select from the following quintuplet of unsolved problems in elementary number theory. This miscellany has been compounded from readers' suggestions taken together with the recent publication entitled *Only Problems, Not Solutions!* by Florentin Smarandache, Xiquan Publishing House, 1991. As well as stimulating readers to respond, this article is intended as a piece of 'market research' to ensure a sequence of popular articles in the future.

Problem I Given a positive integer, n, greater than one, investigate possible solution sets, in PRIME NUMBERS, $(x_1,x_2,x_3....x_n;y)$ of the Diophantine Equation:

$$y = 2.x_1.x_2.x_3....x_n + 1. \text{*********(I)}.$$

e.g. n = 3, 691 = 2.3.5.23 +1, or
647 = 2.17.19 + 1.
n = 4, 571 = 2.3.5.19 +1.

Conjecture For each value of n there exists an infinite number of such solution sets to equation (I).

How are the solutions distributed? Are general formulae possible?

Problem II (Catalan's Conjecture) Given a non-zero integer, k, construct integer solutions (p,q;x,y;), each greater than1, of the Diophantine Equation:

$$x^p - y^q = k. \text{*********(II)}.$$

Conjecture For a given k-value only, a finite number of such solution sets are possible.

Note For the elementary case, k = 1, this was conjectured by JWS Cassels in 1953 and subsequently, 1976, proved by R Tijdeman.

Possible Hint In searching for solutions it would seem appropriate to first decide upon a pair of indeces (p,q) and then investigate acceptable x,y values.

Problem III This problem relates to a general number base, b, where the digit string $a_m a_{m-1}....a_3 a_2 a_1 a_0$ represents the integer:

$$a_0 + a_1.b + a_2.b^2 + a_3.b^3....+ a_{m-1}.b^{m-1} + a_m.b^m.$$

Given b, together with a finite digit string, $c_1c_2c_3...c_n$------(*), design and implement an algorithm to list integers containing the given string (*) which are as follows:

(1) prime, (2) factorial, (3) of the form s^s.

e.g. In the 'usual' decimal representation with b = 10 and (*)—123 we find primes such as 1123, 1231, 8123 etc.

When does (*) occur in s! or s^s? Must it always occur for sufficiently large s? What does the empirical evidence suggest?

Problem IV Given a sequence of integers (x_n), where n = 1,2,3,4... and a digit k, define a SEQUENCE of POSITION FUNCTION, U, thus:

$U_n^{(k)} = U_n^{(k)}(x_n)$ = j if k is the j^{th} (counting from the left) digit of x_n.
= 0 otherwise.

e.g. If x_1=5, x_2=17 & x_3=715 then $U_1^{(7)}$= 0, $U_2^{(7)}$= 2 & $U_3^{(7)}$= 1 because 7 is the first digit in x_3, the second digit in x_2 and does not occur in x_1.

Note When k occurs at more than one position in x_n the function U is multiply-valued. e.g. If $(x_n) = (p_n)$ the sequence of primes then x_{187} = 1117 and so $U_{187}^{(1)}$ = 1,2,3.

Smarandache asks if the sequence of position function, $U_n^{(k)}(x_n)$, generates an infinity of values of type x_n for each k. *Viz* Are there an infinity of k in (i) prime positions in (p_n); (ii) factorial positions in (n!);)iii) 's^s — positions' in (n^n)?

Design and implement an algorithm to input a general integer and to compute/tabulate the associated values of the sequence of position function. Using as input to this algorithm primes, factorials, n^n or other suitable sequences, investigate the Smarandache question as far as your maximum available integer length permits. (This may not generate substantial evidence!)

Problem V As in Problem III let M be an integer in arithmetic base b, the ordered sequence of all distinct digits of M constitute its GENERALISED PERIOD, g(M). This is a subset of (0,1,2,3,...b-1). The NUMBER of g(M), $n_g(M)$, is the number of strings of M such that each contains g(M). The length of the generalised period, $1_g(M)$, is the number of digits in it. Thus if M = 104001144, in some base greater than or equal to 5, g(M)=(0,1,4), $n_g(M)$=2 because 104 and 001144 each contain all the elements of g(M); and because there are three of these $1_g(104001144)$=3.

Questions:

a) Determine n_g & 1_g for primes, factorials and n^n.

b) For a given k, positive integer, are there an infinity of primes (factorials), (n^n) having 1_g=k?

c) Let $a_1,a_2,...a_h$. be distinct digits; are there an infinity of primes (factorials) or (n^n) having these digits as their generalised period?

Attempts at these problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 October 1992.

Review: March 1992 — Pythagoras Rules ABC!

Suggested by CJ of Berkshire, and predated seven years by James Tinsley of Dublin, this topic proved to be extremely popular. Many readers established that the number of RATs having a given hypotenuse, H, depended only upon the multiplicities of the prime factors of H of the form 4n+1. In particular, Mark Cowne of Nottingham with an empirical approach followed by a 'fairly machine independent code' is to be congratulated. Paul Rayer and Jim Duncan each investigated the rectangular parallelepiped and found a dependence on the prime factors of D, not just upon their multiplicities. Nigel Hodges refers to *The American Math Monthly*, June-July 1991, pp505-517 for this topic together with a number of related ones.

The very worthy prizewinner is Roderick PC Forman of 1 Weston's Yard, Eton College, Windsor, Berkshire SL4 6DB. In two dimensions a matrix representation of the algebra was followed by Mathematica version 2 routines run on a NeXTstation. Similar analysis for Pythagorean Quadruples was followed by a study of $c^2=a^2+b^2-kab$ (rational k) in particular with k = 1. Very well done, Roderick!

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.