A positive approach

Two very different examples of Sequences of Positive Integers, A-Numbers and U-Numbers, with Mike Mudge.

Definition A sequence of positive integers consists of a list of such integers, separated by commas. It is read from left to right and is completely defined if there is a rule for generating each number. In some examples it may be possible to generate any given number directly: e.g. Rule. The n^{th} number is 2^n, where clearly the 15^{th} number is 2^{15} = 32768: however in other examples it may be essential to generate either some or all of the preceding numbers first. See examples below.

Sequences may be finite, i.e. there are no further terms defined beyond a certain point, or infinite, i.e. non-terminating. In some cases it is not known whether a given rule yields a finite or infinite sequence.

Sequences are known to be MONOTONIC INCREASING if every term is larger than the term which immediately precedes it. Similar definitions apply to MONOTONIC DECREASING and to NON-INCREASING and NON-DECREASING; in these latter cases successive terms may be equal.

ALIQUOT, A-SEQUENCES

A proper divisor of a positive integer, n, is any positive integer except n itself which exactly divides n. f(n) denotes the sum of these proper divisors of n, thus f(21) = 1 + 3 + 7 = 11.

Now, repeated application of this function to any 'seed' n will generate a sequence, the m^{th} term of which is the result of m applications of this process. For example, using 'seed' 15 we obtain: $f(15) = 1 + 3 + 5 = \underline{9}$, $f^2(15) = f(f(15)) = f(9) = 1 + 3 = \underline{4}$,
$f^3(15) = f(f(f(15))) = f(f(9)) = f(4) = 1 + 2 = \underline{3}$.

E Catalan conjectured, as long ago as 1887, that this sequence is either periodic or stops at the number 1. (Technically we may consider 1 to lead to a period, a trivial repetition of 1..)

There is now a great deal of empirical evidence, together with some heuristic argument, to suggest that some sequences, perhaps even the majority of those with even 'seeds', are of infinite length and not periodic.

It is known that using a 'seed' of 936 yields the following sequence: 936,1794,2238,2250,...74,40,50,43,1 containing 189 terms, the largest of which has 15 digits.

A seed of 138 reaches a maximum of 179931895322 (the 117^{th} term) and then falls to 1 at the 177^{th} term.

The next problem 'seed' seems to be 276 where the 469^{th} term has 45 digits. What happens then?

Problem A Design and implement an algorithm to construct the ALIQUOT, A-SEQUENCE, arising from repeated application of the proper divisor sum function, f, to a given 'seed' n. Using an appropriate choice of seeds investigate the Catalan Conjecture given above.

ULAM, U-SEQUENCES

These sequences, named after Stanislav M Ulam, are generated from two 'seeds', u_1 & u_2, and are continued by including only those integers which can be expressed uniquely (in just one way) as the sum of two distinct earlier members of the sequence. Clearly this provides an example of a sequence where, in order to establish a given term, it is essential to have obtained all the preceding terms in the sequence first.

The FUNDAMENTAL U-SEQUENCE is defined by $u_1 = 1$ & $u_2 = 2$: 1,2,3,4,6,8,11,13,16,18,26,28,36,38,47,48,53... $u_{100} = 690$, $u_{500} = 5685$, $u_{1000} = 12294$.

Clearly this sequence can be generalised by the use of alternative 'seeds', but the questions below will remain unaltered.

Question 1 When is the sum of two consecutive terms of a U-sequence also a member of that sequence?

Clearly $u_1 + u_2 = u_3$ but also $u_{19} + u_{20} = u_{31}$...

Question 2 Which positive integers are not the sum of two terms of a given U-Sequence?

For the fundamental U-Sequence these include 23, 25, 33, 35, 43, 45, 67, 92...

Question 3 Which pairs of consecutive integers are to be found in a given U-Sequence?

For the fundamental U-Sequence these include (1,2), (2,3), (3,4) & (47,48). See the results of P Muller quoted here.

Question 4 Are there arbitrarily large gaps between consecutive terms of a given U-Sequence?

Problem B Design and implement an algorithm to construct the U-Sequence arising from an arbitrary pair of seeds. Use this to address the above questions and also to verify the following result:

- P Muller, 1966. For the first 20000 terms of the fundamental U-Sequence the only pairs of consecutive terms are those given in Question 3 above, but that over 60% of consecutive terms differ by precisely two.

Attempts at the above problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 July 1993.

Any communications received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date. It would be appreciated if such submissions contained a brief description of the hardware used, details of coding, run times and a summary of results obtained; all in a form suitable for publication in *PCW*. Additional comments on the relative levels of interest generated by A- and U- sequences would be appreciated, to provide possible guidelines for future articles in this column.

Please note that material can only be returned if a suitable stamped addressed envelope is provided.

Review of 'Numbers Count' -117- December 1992: The Game of Life

This produced a somewhat disappointing response. Any further reader reaction to both two- and three-dimensional Life Games would be appreciated, while further information relating to three-dimensional Life and supplied by Professor Carter Bays of the University of South Carolina will be forwarded to interested readers free of charge upon receipt of a self addressed A4 envelope with 36p postage.

This month's prizewinner is Gareth Suggett of 34 Bridge Road, Worthing, Sussex BN14 7BX, who was inspired to create a program for three-dimensional Life on his BBC Micro which he is looking forward to speeding up by transfer to a new IBM compatible...

Note of interest:

The 763^{rd}, 764^{th}, 765^{th}, 766^{th}, 767^{th} and 768^{th} digits of PI are all equal to 9. When does a longer run of 9's occur in PI?

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.