Exercises in symmetry

Some problems on a generalised chessboard, proposed by Paul Rayner and presented by Mike Mudge.

The area of investigation this month is due to a regular reader, Paul Rayner of Tonbridge, Kent, and will be expounded using his motivation and philosophy.

Paul had spent 'a frustrating few days looking for an effective algorithm to solve the Chinese Checkers (Solitaire) problem but not finding one [*PCW* March 1993] and, at the same time, reading Michael Field and Martin Golubitsky's excellent book, *Symmetry in Chaos*. I have been reflecting on how symmetry might be used to solve this and similar problems.'

A Solitaire Board has eight symmetries, as does a single square. An 8x8 standard chessboard has four symmetries if the black and white squares are regarded as distinct, but eight symmetries if colour is ignored. However, a 7x7 chessboard always has eight symmetries.

We are invited to consider two different problems: the familiar Eight-Queens problem and the less familiar Knight's Tour, both being extended to a general board size, when the former is renamed the Multi-Queens problem.

Knight's Tour

The aim is to calculate the total number of solutions for each size of board, and to break this down into the number of different tours leading to each possible finishing square. Looking for all possible solutions rules out any heuristic approach to optimise the search for a single solution.

Symmetry about one major axis of the board can easily be incorporated, roughly halving processing times, but the extension to further symmetries is difficult. Paul found the Knight's Tour revealed a single solution on a standard board in 77 seconds (20MHz 80386 computer), but for an exhaustive search was limited to a 6x6 board. Starting from a corner square, symmetry about the diagonal axis passing through that square is readily included.

Thus, only 10 of the 64 squares on an 8x8 (6 out of 36 on a 6x6) need to be considered as starting squares, the remainder being symmetrically equivalent. Symmetry does not seem to help much when tracing each to. There is only one (symmetrical) way in and out of each corner square; the generalisation of this suggests the use of: 'For any vacant square there must be, at each stage of a successful tour, at least two vacant squares a knight's move away unless the knight is about to move onto that square.'

Now assume the origin to be at the top left corner of the board and further that square (1,1) is white. The knight will, clearly, move to black and white squares alternately.

(i) If the length of a side of the board is an odd number of squares, then tours covering all squares will start and end on the same colour. Hence, since there is one more white square than there are black squares, and the knight will make an even number of moves, successful tours must start and finish on white squares.

(ii) If the board has an even number of squares, the tour will end on the opposite colour square to that on which it starts.

(iii) For a 5x5 board, a set of tours can be found starting from any corner square that end up on each of the remaining white squares. Starting from any other white square, tours end only on the four corner squares. In other words, a tour must either start or finish on a corner square.

Can these statements be extended, either theoretically or experimentally, to other board sizes?

(iv) Experimental evidence suggests that the first statement in (iii) translates to a 6x6 board, with a set of tours ending on each of the black squares. Of the 524486 different tours starting from square (1,1) 19724 could return to the starting square by making a 36th move.

Can these numerical results be confirmed, extended and explained?

Multi-Queens

Here the problem is to place N-Queens on an NxN board so that no Queen is attacking any other Queen. The number of ways in which this can be done is denoted by F(N) and some data, with computing time on the 20MHz 80386 displayed in Fig 1. Can these numerical results be confirmed, extended and explained?

N	8	12	16	20
F(N)	92	14200	14772512	???
Time	0.1sec	47.0sec	8hrs +	2yrs +?

Fig 1

Attempts at the analysis of the above problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 August 1993. It would be appreciated if such submissions contained a brief description of the hardware used, details of coding, run times and a summary of results obtained... together with any supporting algebraic theory!... all in a form suitable for publication in *PCW*. Additionally, comment upon, or references to, any other problems based upon gameboards in which symmetry is thought to play an important role, would be of considerable interest.

Review, *PCW* January 1993

For personal reasons this proved to be a rerun of Numbers Count -92- *PCW* October 1990, but was, surprisingly, more popular the second time around.

The underlying difficulties centred around (a) a measure of efficiency and (b) the statistical distribution of file lengths for a typical user — whoever that may be! Something vaguely Poisson-like or even Exponential appears to be more realistic than the Uniform Distribution assumed by some readers.

Responses ranged from 'Data is too precious to scrooge on floppies', accompanied by a promise of a detailed response which did not arrive; to extensive algorithms in C for the Borland C++ version 2.0 compiler by David Broughton. Nick Weatherhead worked in Commodore 64 Basic, one of several using a Uniform (Rectangular Distribution) Distribution for file lengths.

However, this month's prizewinner is Robert Barbiaux, of 17 rue General Thys, 1050 Bruxelles, Belgium, who classified this as a typical 'covering' problem and used a backtracking algorithm in Turbo Pascal, very well documented and extensively tested on a Toshiba 5200 computer. Well done, Robert. Details from Brussels.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.