A full-house of prime numbers

A small? investigation involving Prime Numbers, together with a revisit to Euler's Totient Function.

This concept is due to Ram Nair of Havant, Hampshire who, perhaps somewhat optimistically, refers to it as a small investigation.

Definitions P_n denotes the n^{th} prime number commencing with p_1 = 2....thus p_{9517} = 99131.

If (10*N + 1), (10*N + 3), (10*N + 7) and (10*N + 9) are all prime numbers then the 'DECAD' of ten numbers starting with 10*N is termed A FULL HOUSE.

Ram has found all the FULL HOUSES below 100000, there are 35 of these. The values of n and p_n (the first prime number in each) include those shown in Fig 1.

Ram asks the question 'whether a more general definition of a FULL HOUSE as ten consecutive integers enclosing four prime numbers might be more interesting ... a kind of generalisation of TWIN PRIMES?'

Note: For a study of TWIN PRIMES readers are referred to PCW July 1984.

Problem 1 Investigate the distribution of FULL HOUSES, using both the definition consistent with Fig 1 and the alternative suggested above. If sufficient empirical data can be obtained use a graphical or other technique to predict the number of such structures less than 10^m, for large m, and hopefully other readers or sources will have data to verify these predictions.

Euler's Totient Function (Revisited)

This approach has been suggested by Mike Leigh of Salford. Many thanks to him for a most interesting proposal. Euler's phi-function, or Euler's totient function (named after the great Swiss mathematician, Leonhard Euler (1701-1783)) is defined to be the number of positive integers less than n and relatively prime to n (i.e. not sharing a common factor other than 1). Further, phi(1) is defined to be 1, e.g. phi(15) = 8, the eight numbers being 1,2,4,7,8,11,13 & 14; phi(p) = p-1 when p is prime, phi(405769) = 321048, phi(3438955969) = 3401526000 may be used to test any computer routine for the evaluation of the phi-function.

Number of FULL HOUSE	1	2	3	...	25	26	27···	35
n	5	26	43	...	2785	3410	3718 ..	9517
p_n	11	101	191	...	25301	31721	34841 ..	99131

Fig 1

Now for x greater than 1 we have phi(x) less than x hence iteration of the phi-function for any x (positive integer argument) must eventually yield 1. Let H(x), the HEIGHT of x, be defined as the number of iterations required to reach 1. It is 'easy' to prove (using mathematical induction) that:

for x & y both even H(xy) = H(x) + H(y) - 1 otherwise

H(xy) = H(x) + H(y) and further that H(x) is bounded by $\log_2(x)$ on one side and $\log_3(x)$ on the other side.

Problem 2 What happens to the average value? i.e. Does (H(1)+H(2)+H(3)...+H(n))/(nlog(n)) tend to a fixed limit as n tends to infinity? Investigate this limit for n as large as is feasible.

The most interesting result of Mike Leigh is that the phi-function shows similar behaviour:

If x & y are coprime then phi(xy) = phi(x) * phi(y) else if x&y have a highest common factor h then phi(xy) = phi(x)*phi(y)*h/phi((h). The proof of this is left as an 'exercise for the readers'; some (many) cases may be satisfied with numerical verification in a number of cases. Mike calls such a function A TOWERING MULTIPLICATIVE FUNCTION. Example $x=1100=2^2 5^2 11$ $phi(x)=2^4 5^2$, $phi(phi(x))=2^5 5$, *3 $phi(x)=2^6$, 5 $phi(x)=2^4$ etc *where n phi(x) denotes the n^{th} iterate of the phi function.

The name TOWERING arising from the fact that the power of each prime gradually builds up and then dies down again as the iteration proceeds.

Now we define the function $L_{phi}(x)$ to be the sum of phi and all** its iterates (assuming such a sum exists) each being operated on by the COUNTING FUNCTION, MU defined for example by MU_2 = 1 if x is divisible by 2 and zero otherwise; similarly for MU_p for any prime p.

Thus $L_{mu}(x)$ = $mu(x)^{**}$+mu(phi(x))+mu(phi(phi(x))) +...

**Including the zeroeth!

Problem 3 (1) Does the sum of $L_p(x)$ (i.e. over all n less than or equal to x) when divided by xlog(x) tend to a limit as x tends to infinity: where phi is the Euler Totient Function and L_p implies the use of the counting function MU_p?

(2) If phi is redefined as phi(q) = $(q-1)^2$ for each prime q and the log function in the denominator is squared, does a limit exist?

Mike states that calculations which he did on a Casio programmable 15 years ago (and now lost!) suggested answers yes to both questions.

Attempts at some or all of the above problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 September 1993. Communications received will be judged, using suitable subjective criteria, and a prize will be awarded by *PCW* to the 'best' contribution arriving by the closing date. Comments on the relative merits of the two research areas presented this month would also be appreciated.

Review of 'Numbers Count -118- February 1993: a revisit to The Florentin Smarandache Function

This produced a number of 'quite powerful' responses. As a note of related interest, the latest publication of Fl.Smarandache is 'A Numerical Function in Congruence Theory', *Libertas Mathematica* (American Romanian Academy of Arts and Science) vol 12, 1992, pp 181-185, Arlington, Texas.

Pal Gronas of Norway submitted theoretical results on both problems 0 & (v). However, the clear winner this month is a former regular respondent, now retired, Henry Ibstedt, Glimminge 2036, 280 60 Broby, Sweden. Henry used a dtk-computer with 486/33MHz processor in Borland's Turbo Basic. S(n) upto 10^6 took 2hr 50min. He completed a great deal of work on all problems except (vi): details of numerical results and conclusions available from Henry or myself to interested readers. What about problem (vi)?

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.