Class act

A simple classification of (all) problems in arithmetic, presented by Mike Mudge.

(The distinction between the solution of a problem and the accumulation of empirical evidence relating to it.)

Readers' attention is first drawn to the publication, *A Selection of Problems in the Theory of Numbers* by W Sierpinski, Pergamon Press, 1964. The final chapter is entitled 'One Hundred Elementary but Difficult Problems in Arithmetic'... Before considering any specific problem it is appropriate to attempt a simple classification. Problems of THE FIRST KIND are those for which it is known how to obtain the complete solution; the only difficulty being that, even with the assistance of the biggest computers which exist at present we are not able to perform all of the necessary computations.

Problems of THE SECOND KIND are those for which no method is known that could lead to a complete solution, even after the most extremely long and tiresome calculations that may well exceed our present computational capacities. Now, for these problems a great deal of time and effort can be spent acquiring evidence which may suggest the nature of the solution, i.e. empirical evidence, but it needs to be clearly understood that no amount of such evidence constitutes a solution.

Problems of THE THIRD KIND, which will not concern us in this article, are those which have not yet been classified under TWO/ONE above!

(Those which have been solved, although we may not be aware of the solution, are not regarded as problems any more.)

In 1964, W Sierpinski's examples of THE FIRST KIND include:

a) Find all the natural divisors (factors) of $2^{101} - 1$. This is clearly no longer a problem. What is its solution? However, a similar problem could be readily formulated by replacing 101 by a suitably large integer.

b) To write the number 10 as the sum of a finite number of distinct rational numbers (fractions) with numerator 1. It is known that there are at least 12,366 of these... Is this still a problem? Once again, the increase of 10 to 10^n for a suitably large n would certainly yield a PROBLEM of the FIRST KIND.

Note that PROBLEMS of the SECOND KIND may cease to be problems, clearly not as a result of any amount of computation but due to advances in the theoretical studies which lay behind them. For example, in 1845 J. Bertrand made the conjecture that if n is a natural number greater than 1 then there will always be at least one prime number between n and 2n. At that stage a PROBLEM of the SECOND KIND, since any computation could only reach a finite n. However, theoretical proofs arose, initially due to Chebyshev and subsequently Sierpinski.

c) Is every natural number the sum of nineteen fourth powers of integers? On the face of it, this appears to be a problem of the SECOND KIND due to the occurence of the term 'every natural number'. However, FC Auluck proved that it is true for every integer greater than $10^{10^{89}}$ and so computation up to that value would complete the proof. But it is clearly not feasible...

Among the simplest PROBLEMS of the SECOND KIND Sierpinski asks:

d) Does the digit 1 occur infinitely many times in the decimal expansion of the square root of 2? Explain clearly why this is of the second kind and provide empirical evidence for its solution.

e) Does the equation $x^3+y^3+z^3-t^3 = 1$ have infinitely many distinct solutions in positive integers x,y,z,t? Is this a problem and, if so, of what kind and why?

f) Obtain all the integer solutions of the equation $x^3-y^2 = 18$ in integers. (It has been proved that such solutions are finite in number; so what KIND of problem is this and why?)

g) Find all prime numbers of the form n^n+1, $n^{n^n}+1$ etc. What kind of problem is this and why?

Attempts at classification and investigation of some, or all, of the above problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 June 1994. Any communications received will be awarded, by *PCW*, to the 'best' contribution arriving by the closing date. It would be appreciated if such submissions contained a brief description of the hardware used, details of coding, run times and a summary of results obtained, all in a form suitable for publication in *PCW*.

Additionally, readers' comments on the general, or specific, nature of this month's column together with references to any recent work on the types of problems posed, would be most valuable.

Review of Numbers Count -127-, *PCW* November 1993: Repunits, Repetends and Something Different involving the number 17

A number of very high quality responses were generated by this month's topics. Paul Rayner factorised all repunits up to R_{72}, with the exception of R_{71} (no result after 60 hours), using the Elliptic Curve factorisation routine in UBASIC on a 20MHz Vig III using an 80386 + 80387 co-processor. He also identified 29928 Smith Numbers below 10^6. David Broughton found a remarkably picturesque way of describing the 17- problem, involving tree planting: 'A man has a linear strip of land in which he wishes to plant trees, one for each of his children, as yet unborn. He wants each tree to be planted in such a way that however many children he finally has, each tree will grow in a section which divides the whole plot equally among all his children. Thus as each child is born, new boundaries are drawn to divide the land equally and just one more tree is planted for that child in a plot which does not contain a tree. In the remaining plots must exist one and only one tree.' The required results were then obtained in Borland's C++ version 2 compiler on a 386 CPU running at 33MHz.

Among other creditworthy submissions was one from Gareth Suggett involving the classification of squares according to their n-digit endings; and one from Henry Ibstedt with a simulation-type approach to the 17-problem.

However, the worthy prizewinner this month is Nigel Hodges of 103 Grange Road, Tuffley, Gloucester GL4 0PT, whose results include a proof that the smallest repunit which is a k^{th} power, for k greater than or equal to 2, must have prime length. Nigel also gives the reference to Harvey Dubner, *Mathematics of Computation*, vol 61, October 1993, pp927-930 Generalised Repunit Primes, which looks very interesting.

Mike Mudge welcomes readers' correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.