In the prime of life

The fascinating but easily defined world of factorials, sub-factorials and primorials, presented by Mike Mudge.

Definitions

(1) A PRIME NUMBER is a positive integer which is only divisible by itself and unity (one). Thus the infinite sequence of prime numbers begins 2,3,5,7,11,13,... Note: For a variety of proofs that there are indeed infinitely many prime numbers, see, for example, *The Book of Prime Number Records*, Paulo Ribenboim, Springer-Verlag, 1988. Ch. 1.

(2) The FACTORIAL FUNCTION with argument n, for positive integer n, written n! is defined as the product of all of the positive integers upto and including n. Thus the sequence of values of n! begins 1,2,6,24,120,720,... We may, for what at this stage are reasons of convenience, define 0! = 1.

(3) The SUB-FACTORIAL FUNCTION with argument n, for positive integer n, is written !n and defined thus: !n = n! (1 - 1/1! + 1/2! - 1/3..1/n!)

(4) The PRIMORIAL FUNCTION is defined ..only for argument p, a PRIME NUMBER.. as the product of all prime numbers less than or equal to p. Thus the sequence of values of PRIMORIAL p, or p*, begins:2,6,30,210,2310,.....

Ribenboim poses the following unsolved problems regarding p*:

Are there infinitely many p such that p* + 1 is PRIME?

Are there infinitely many p such that p* + 1 is COMPOSITE? (i.e. non-prime.) These are too difficult for us to address here, but we may confirm (and extend) the following evidence:

13649* + 1 is probably the largest known prime of the form p* + 1; it has 5862 digits. It is also known that p* + 1 is prime for p = 2,3,5,7,11,31,379,1019,1021 & 2657 and composite for all other p less than 11213. It is also prime for p = 11549, 4787, 4547 & 3229.

Problem A Confirm as much as possible of the above data and repeat the exercise for p* - 1. Check data, from *Curious and Interesting Numbers* by David Wells, Pengiun 1986.

Primality for p = 3,5,11,13,41,89,317,991,1873,2053 and no other value below 2377.

'Most integers have very few distinct prime factors.' Primorials are clearly exceptional in this respect...

Problem B Calculate the average number of distinct prime factors of all integers less than or equal to a given N. Repeat this process with the primorials omitted and attempt to predict the behaviour of this average, in the limit as N becomes infinite.

Problem C p* = 510510 when p = 17. This is also the product of four consecutive Fibonacci Numbers**: 13x21x34x55. Are there any other p* with this property? i.e. which are the products of (consecutive) Fibonacci Numbers.

**Defined by $F_0 = F_1 = 1$: $F_{n+1} = F_n + F_{n-1}$ for n greater than zero. Furthermore, 17* = 714 x 715, are there other primorials which are the products of consecutive integers?

Returning now to SUB-Factorials.

Problem D 148349 = !1 + !4 + !8 + !3 + !4 + !9 is known to be the only integer equal to the sum of the sub-factorials of its digits when arithmetic is expressed radix ten; determine numbers having this property in other arithmetic bases.

Historical note: (reference David Wells, *Curious and Interesting Numbers*.) Nicolaus Bernoulli first considered the problem of n letters written to n different addresses and n matching envelopes. !n gives the number of ways in which the letters can be placed, one to each envelope, such that every one goes to a wrong address.

Problem E 9 = !3 viz. 1! + 2! + 3! = 9 = $1^3 + 2^3$. Determine the distribution of sub-factorial numbers which are expressible as the sum of two (or more) cubes (or other powers) of consecutive integers.

In conclusion, returning to FACTORIALS.

Problem F 40585 = 4! + 0! + 5! + 8! + 5! Investigate the existence of other integers which are equal to the sum of the factorials of their digits, both in arithmetic base ten and in other bases.

Problem G 3628800 = 10! = 6!x7! = 3! x 5! x 7! Investigate the existence of other integers whose factorial is expressible as the product of other (consecutive) factorials.

Problem H — no computer required! For what n are the number of digits in (i)n!, (ii) !n, equal to n,2n,3n...

Attempts at some, or all, of the above problems may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 July 1994. Any communications received will be judged, using suitable subjective criteria, and a prize will be awarded, by *PCW*, to the 'best' contribution arriving by the closing date. Readers' comments on the general or specific nature of this month's column together with references to any recent work on the problems posed would be most valuable.

Review of 'Numbers Count' -128- *PCW* December 1993: Cunningham Chains (Sophie Germain Primes) and Prime-Valued Polynomials

These topics produced a number of detailed responses including: Gareth Suggett, who located all chains of length 5 or more with starting number less than 10^5, those of length 6 or more upto 10^6, those of length 7 or more upto $2x10^7$, reaching the first length 8 chain. This was followed by chains of the second kind upto length 7 with starting values less than $5x10^6$. Gareth corrects RK Guy, *Unsolved Problems in Number Theory*, p13, where 67651 does not give a chain of length 7 but, in fact, of length 4.

Henry Ibstedt also corrected RK Guy and followed an extensive search for Cunningham Chains with a study of Prime-Valued Quadratics, Polynomials of Degree 1 and a Cubic Polynomials-500 Race with 'winner' $2x^3$ - 485x2 + 40017x - 1000033 with V = 241.

Tiger Redman carried out sterling work on his BBC Model B to find 11,7-chains (2n+1) and one 8-chain starting at 19099919 also 17, 7-chains (2n-1) and one 8-chain starting at 15514861. Additionally, Tiger's results include (2n+13) with a 7-chain starting at 17, a 9-chain starting at 3467 and an 8-chain starting at 60077; also (3n-4) with a 9-chain starting at 1087. However, the winner this month is Paul Rayner of Remingtons Farmhouse, Lamberhurst Road, Horsmonden, Kent TN12 8LP, who used UBASIC on a 20MHz Vig III to search n = 1(1)25 and m = -30(1)+30 with initial prime values upto 1000000 in about 30 hours for problem 1; together with some investigation of problem 2. Details on request.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.