# A classic case

## Extensible sets, congruent numbers and a miscellany from the Archimedeans, presented by Mike Mudge.

The combination of two classical research areas with a selection of problems from The Cambridge University Mathematical Society (The Archimedeans) is intended to provide something attractive to every Numbers Count reader. Remember that all submissions are considered for the prize and that many lasting friendships have been established through this column!

TOPIC 1 Sets in which $xy + k$ is Always a Square.
See, for example, Ezra Brown, Mathematics of Computation, vol. 45, no. 172, October 1985, pp613-620.

A $P_k$-set of size n is a set of n distinct positive whole numbers $(x_1,x_2,...x_n)$ such that $x_ix_j + k$ is a perfect square whenever i is not equal to j. A $P_k$-set is said to be EXTENSIBLE if there exists an integer y such that $(x_1,x_2,...x_n,y)$ is still a $P_k$-set. e.g. The $P_1$-set (1,3,8,120) cannot be extended.

(1,2,5) is a $P_{-1}$-set of size 3,

(1,79,98) is a $P_2$-set of size 3 while (51,208,465,19732328) is a $P_1$-set of size 4.

In the above reference, the following theorems are proved:
(I) If k is congruent to 2 modulo 4 i.e. $k = 2 + 4m$ for some integer m, then there does not exist a $P_k$ - set of size 4.
(II) If k is congruent to 5 modulo 8 then there does not exist a $P_k$ - set of size 4 with an odd $x_j$ or with some $x_j$ a multiple of 4.
(III) The following $P_{-1}$- sets cannot be extended:
(a) $(n^2+1,(n+1)^2+1,(2n+1)^2+4)$ if n is not a multiple of 4
(b) (17,26,85)
(c) $(2,2n^2+2n+1,2n^2+6n+5)$ if n is congruent to 1 modulo 4
(d) (1,2,5)

Recall that this last set is a $P_{-1}$- set because:

$1x2 - 1 = 1^2$, $1x5 - 1 += 2^2$ and $2x5 - 1 = 3^2$

**PROBLEM 1** Design and implement a computer program to obtain $P_k$- sets of size s for given (sufficiently small) k & s.

As each set is found, investigate possible EXTENSIBILITY, summarise the findings in such a way as to associate them, if possible, with the above theorems and to provide empirical evidence for further general results.

**PROBLEM 1*** What happens if topic is redefined using 'Always a Cube, Fourth Power etc'?

TOPIC II Congruent Numbers
See, for example, Ronald Alter and Thaddeus B Curtz, Mathematics of Computation, vol. 28, no. 125, January 1974, pp303-305.

An integer, a, is called a CONGRUENT NUMBER iff (if and only if) there are positive integer solutions to the pair of equations $x^2 + ay^2 = z^2$ and $x^2 - ay^2 = t^2$, such solutions consisting of values for x, y, z & t.

In the above reference, the Square-free Congruent numbers less than 1000 are tabulated: 5,6,7,13,14,.....987,995,998.

There follows a conjecture that if n is congruent to 5, 6 or 7 modulo 8 i.e. leaves a remainder of 5, 6 or 7 when divided by 8, then it is a CONGRUENT NUMBER. The first unsettled cases at that time being 103, 127 and 133.

**PROBLEM II** Design and implement a program to identify CONGRUENT NUMBERS, according to the above definition, confirm (or reject!) the conjecture of Alter and Curtz: investigate an algrebraic approach.

**PROBLEM II*** What happens if SUPER-CONGRUENT NUMBERS of DEGREE -n are defined by requiring positive integer solutions of the pair of equations $x^n + Ay^n = z^n$ and $x^n - Ay^n = t^n$ where n = 3,4,5...?

I am not aware of any investigation of this problem. It is therefore conceivable that any results obtained may be considered for publication.

**The Archimedeans**
About ten years ago The Cambridge University Mathematical Society, known as The Archimedeans, had several publications including Eureka and QARCH. What is the present situation of the Society? Maxim Tingley, the president at 13/7/84, invited publication of a sample of problems; readers interested in subscribing to QARCH should contact the Editor at The Arts School, Bene't Street, Cambridge CB2 3PY (last known address) for further details.
'Typical' problems:
(A) For what values of n is $n^4 + 4^n$ prime? When is $x^y + y^x$ prime?
(B) Find the general rectangular prism of whose edges, face-diagonals and body-diagonals are integers. In the event of not being able to solve this problem, find a size which such a prism (if it exists) must exceed.
(C) In Pascal's triangle, having rows 1,1;1,2,1;1,3,3,1;1,4,6,4,1;1,5,10,10,5,1; ....the number 3003 occurs 8 times besides the trivial case of 1. Does any other number occur this often?

Response to some, or all, of the problems posed above may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel (0994) 231121, to arrive by 1 November 1994. Any communications received will be judged, using suitable subjective criteria, and a prize of £25 in the form of a book token will be awarded by Mike Mudge to the 'best' contribution arriving by the closing date. It would be appreciated if such submissions contained a brief description of the hardware used, details of coding, run times and a brief summary of the results obtained; all in a form suitable for publication in *PCW* — if and when space allows.

Please note that material can only be returned if a suitable stamped addressed envelope is enclosed.

**Review of Numbers Count -123- Interval Analysis, *PCW* April 1994**
A disappointing response to what is still considered to be a very important subject area in applied computing. However, important points were raised. Alan Cox, and others, observe that experimental errors are not usually uniformly distributed and question the need for carrying an internal value. The fact that the laws of interval arithmetic do not replicate the rules for combining uniformly distributed variates was also noted. Roger Bamkin of Derby has an interest in modelling engineering information and is aware of the difficulties associated with such 'imprecise numbers'. The prizewinner is Gareth Suggett of 34 Bridge Road, Worthing, West Sussex BN14 7BX, with a serious attempt at the quadratic interval algorithm, which revealed the problem of intervals containing zero. Interval analysis is definitely here to stay... interested readers are directed to Eldon Hansen's work on Interval Arithmetic in Matrix Computations.

Mike Mudge welcomes correspondence on any subject within the areas of number theory and computational mathematics. Particularly welcome are suggestions for future Numbers Count articles.