1729 and all that

A remarkable discovery in the field of Diophantine Equations, presented here by Mike Mudge.

In November 1989 this column noted that "The Least Positive Integer Solution of the Diophantine Equation

$s = x^3+y^3 = z^3+w^3 = u^3+v^3 = m^3+n^3$

had been computed by E. ROSENSTIEL, J.A. DARDIS and C.R. ROSENSTIEL and that [Fig 1] is the least positive integer which can be represented as the sum of two cubes in four different ways.

In October 1994 John Dardis wrote: "I now have the value of the first quintuple that I have been searching for (pretty well 24 hours a day) since June 1993.

$$6963{,}472{,}309{,}248 = 2421^3 + 19083^3 = 5436^3 + 18948^3 = 10200^3 + 18072^3 = 13322^3 + 16630^3$$

Fig. 1

"The least solution in distinct positive integers of the Diophantine equation

$s = x^3+y^3 = z^3+w^3 = u^3+v^3 = m^3+n^3 = a^3+b^3$

is shown in [Fig 2].

s =	48 988 659 276 962 496
x =	38 787
z =	107 839
u =	205 292
m =	221 424
a =	231 518
y =	365 757
w =	362 753
v =	342 952
n =	336 588
b =	331 954

Fig. 2

"This latest solution was computed using the same method as was used for the earlier solution, but on a 25MHz 386 computer with 12Mb of RAM and a 387SX co-processor. The program, which was written in Turbo Pascal, used 8Mb of RAM as expanded memory. This gave 512 16kb pages, each of which could store up to 2000 8-byte integers (of type 'comp'). The expanded memory support functions were taken from Michael Tischer's book 'Turbo Pascal Internals'.

"The essence of the method used was to calculate all the possible sums of pairs of cubes within a particular range and then to sort them and check for repeating values. The performance was found to be heavily dependent on very careful tuning to avoid wasteful calculation and on extensive testing of the sort routine to ensure maximum performance."

He concluded this final report thus: The first quintuplet came at the 147th quad, the 16317th triple (counting duplicates) and after about 4,664,989 doubles.

This study is clearly open ended with extensions to six, seven etc. pairs of cubes or to pairs of higher powers or indeed to triplets, quadruplets etc. of cubes and/or higher powers.

PROBLEM R(osenstiel)

Investigate the determination of the smallest positive integer that can be expressed as the sum of 'm' n-TH powers of positive integers in p-distinct ways. (Will some results will be obtained with less computing effort than used in the above result?)

AUTOMORPHIC NUMBERS

This topic has been suggested by Alan Cox, but is also stimulated by reading The Penguin Dictionary of Curious and Interesting Numbers by David Wells.

Definitions A positive integer is said to be AUTOMORPHIC if it is replicated by the final digit sequence in its square. e.g. 625 x 625 = 390<u>625</u>.

A positive integer is said to be TRIMORPHIC if it is replicated by the final digit sequence of its cube. e.g. 49 x 49 x 49 = 117<u>649</u>. So we could define an n-morphic positive integer that is replicated in the final digit sequence of its n^{th} power.

A tri-automorphic number is a positive integer which is replicated in the final digit sequence of three times its square. e.g. 3 x 6667 x 6667 = 133346<u>667</u>.

Hence an m-n-morphic number would be identified by looking at the least significant part of m times its n^{th} power.

Wells asserts that there are three tri-automorphic numbers for any given number of digits...

Problem A_1. Find these triples for any given number length, L-digits.

How does this result generalise for n-morphs?

How does this result translate to arithmetic in number bases other than ten? i.e. What about binary, octal or hexadecimal?

Problem A_2. Wells states that 1787109376 is one of only two 10-digit automorphic numbers: find the other one. Investigate the number of m-n-morphic numbers with a given number of digits, N say. Note: since the property in question depends upon the least significant N-digits, a great deal of the computation of m times the n^{th} power is irrelevant and hopefully will be omitted.

Responses to any of the problems R & A above, together with comments on the computation of Rosenstiel *et al*, may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St Clears, Carmarthen, Dyfed SA33 4AQ, tel 0994 213312, to arrive by 1st April 1995. Any solutions or attempted solutions received will be judged, using suitable subjective criteria, and a prize in the form of a £25 book token, or equivalent overseas voucher, will be awarded by Mike Mudge to the "best"' contribution arriving by the closing date. Such contributions should contain a brief description of the hardware used, details of coding, run times and a summary of the results obtained; all in a form suitable for publication in *PCW*. Additionally, readers' comments upon the general or specific nature of this month's column would be most valuable, in particular references to any recent work (either published or unpublished) in the subject areas covered.

Please note that material can only be returned if a suitable stamped addressed envelope is enclosed.

FEEDBACK — both POSITIVE and NEGATIVE

Readers' requests that a full mathematical type font be made available for this column have been passed on to the editor!

Review of Numbers Count -136- August 1994 — The Final Frontier? — together with a set of REPUNIT RIDDLES

These two topics generated a near record postbag, if only I knew why!

Dealing first with the REPUNIT RIDDLES: r_1 is satisfied by $R_1R_{100}....R_9R_{92}$. r_2 is found to be satisfied by $R_{33585143906}$ because the period length of 183263 is, in fact, 183262 i.e. the recurrence of its reciprocal as a decimal. Interest note: if a printer prints 1400 lines each of 132 characters per minute it would take four months of continuous operation to generate this REPUNIT.

r_3 requires REPUNIT ≡ 12345679 the sum being R_9.

r_4 has two solutions $212/606 = 0.\overline{3498}$ or $242/303 = 0.\overline{7986}$.

r_5 locates a 2 in the 37th decimal place.

The shorter string in r_6 caused some difficulty but Henry Ibstedt, and others, discovered $2^4.3^2.5.7.(59^*) + n$ where 59* denotes the product of all the primes less than or equal to 59, also n runs from 0 to 60. Among readers attacking only these problems mention is made of Robert Newmark, Henry Ibstedt, Eddie Kent and Paul Leyland. The latter introduced The Cunningham Project, "it is probably the longest running computational effort in history and has accumulated many mips-millenia of CPU power in that time." i.e. since the 1920s. Here the object is to factorise $10^n \pm 1$, in general $b^n \pm 1$.

Paul has submitted many such results and also has an interest in factorising $n! \pm 1$. Further details and contact address from the author. (Mike M.).

Among contributors addressing only the compression of data problem: Stefan Smith is currently working on such a system for use with a textual database system. Readers are asked to look out for the lifeBASE Project and the English Language Compression System (ELC), the idea being to produce a standard dictionary to reduce text file size. Expected release date July 1995.

P.R. Luck at first thought the entire article to be a spoof, recalling that a couple of years ago there were articles about a company that claimed it could achieve enormous rates of compression, in a lot of technical journals "that should have known better." He argues the answer, NO (assuming a finite set of operators) as each formula can only represent one number (any operator generating more than one answer, such as a square root, will not be useful since the answer will not be unique), and the formula in question, stripped of its operators, will require another formula to represent it. One can only guess at the patterns of formulae/numbers involved as one chains from one formula/number to the next, but it is certain that the numbers cannot all be smaller than each other. Mat Newman, on the other hand, establishes the answer, YES, beginning with the apparently facetious answer "define the symbol 'X' to represent the long number in question" and going on to distinguish between "encoding" and "compression". Mat also lightens the atmosphere with the result that Hallowe'en and Christmas are identical! 31OCT(al) = 25DEC(imal).

Now to the hardworking individuals who considered both subject areas. Nigel Hodges cites Comtemporary Mathematics, vol.22, by The American Mathematical Society. Brillhart, Lehmer, Selfridge, Tuckerman & Wagstaff Factorisation of $b^n \pm 1$ for selected b up to high powers. In particular $10^n + 1$ is completely factored for n=1 to 121 *except* for n = 106 and 109. Any offers to fill these gaps?

Gareth Suggett observed that the compactness of any representation of a group of numbers depends on (a) how many numbers there are in the group, and (b) the size of the "alphabet" in which the strings representing the numbers are written. Thus if one has N numbers to represent and s symbols available, then the general number will need a string whose length is of the order of $\log_s(N)$ to represent it. The most compact representation will have s = N.

George Sassoon provided a very extensive reply, referring to The Cunningham Project masterminded by Prof. Sam Wagstaff, Dept. of Computer Sciences, Purdue University, West Lafayette, Indiana 47907, USA. George uses Kida's PPMPQS programs which accompany UBASIC and suggests that interested readers should contact Prof. Wagstaff; he introduces The Hilbert Curve as a succession of right-angled turns which at a high enough order will fill any n-dimensional space, and so the space can be represented as a single line of data regardless of its dimensionality and hence the Hilbert curve is generally more efficient at storing an image than a conventional scan.

Eamonn Maher of Roscrea, Co. Tipperary, supplied a publication, Randomness and OMEGA, inspired by Chaitlin of the Bell Labs and leaning heavily on Gödel's Theorem, copies available on request, having relevance to the data compression problem.

However, the very worthy prizewinner is Malcolm Gray, of 19 Hadrian Way, Sandiway, Northwich, Cheshire CW8 2JR, using Borland C++ 3.1 on a 486DX/33 for the larger REPUNIT calculations and establishing that a language of (0,1,2,3,4,5,6,7,8,9,+,-,e,*) enables $1.4^{-3/4n}$ numbers can be coded in less than a quarter of the digits. This tends to zero very rapidly, so for large numbers very few can be so represented and the conclusion is that a drastic improvement is only possible in a relatively few cases.

PCW Contributions Welcome

Mike Mudge welcomes readers' correspondence on any subject within the areas of number theory and computational mathematics, together with suggested subject areas and/or specific problems for future Numbers Count articles.