Binomial Coefficients

Beginners Lesson Three — an introduction to Binomial Coefficients; plus a pot-pourri from recent correspondence. With Mike Mudge.

BEGINNERS LESSON THREE

Given any positive integer n, the function FACTORIAL of n, written n!, is defined as the product of all the positive integers upto and including n. Thus 6! = 1x2x3x4x5x6 = 720.

Note that for convenience 0! is DEFINED to be 1.

Given a positive integer n and a non-negative integer r less than or equal to n the BINOMIAL COEFFICIENT $\binom{n}{r}$, or ${}_nC_r$, is defined to be

$$\frac{n!}{r!\,(n-r)!} \quad \text{Thus} \quad {}_7C_4 = \frac{7!}{4!\,3!}$$

These coefficients have applications in the mathematical topics of The Binomial Theorem and Combinatorics, however these will not be required for the following material to be comprehensible.

It is easily shown, from the above definition, that:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

thus

${}_{10}C_5 = {}_9C_4 + {}_9C_5$

and the scheme known as Pascal's Triangle for the generation of such coefficients is revealed:

```
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
1......................1
```

Here every element, except the bounding diagonals of 1's, is formed as the sum of its two nearest neighbours in the previous row.

Clearly these coefficients can be evaluated from the original definition, from the Pascal's Triangle Algorithm or indeed from the equivalent algebraic result

$${}_nC_r = \frac{n.(n-1).(n-2).....(n-r+1)}{r.(r-1).(r-2).....1}$$

Problem (A). Implement the above three algorithms for ${}_nC_r$, check each using ${}_{12}C_5$ = 792, consider the ultimate limitations of each program by attempting, initially, ${}_{200}C_5$ which is approximately 2.53565×10^9.

How do the ultimate limitations, in the sense of largest n & r values which can be handled exactly, compare with run-times? Could the latter be important in the evaluation of many such coefficients, see Problem (B) which follows?

Fig 1 shows a sample program, in BASIC, due to S. Balachandra Rao & C.K. Shantha...Numerical Methods..Sangram Books Ltd. 1992, £9.95; for the evaluation of Binomial Coefficients.

Problem (B). With reference to the properties of Pascal's Triangle, Pascal himself wrote: "I leave out many more than I include; it is extraordinary how fertile in properties this is. Everyone can try his hand."

Verify that the entries in rwo p, except the units, are divisible by p if and only if p is prime. Verify also that the shallow diagonals 1,1-1,1-2,1-3-1,1-4-3,1-5-6-1...etc. sum to the Fibonacci sequence 1,1,2,3,5,8,13,..etc. (Each entry, apart from the first two 1's being the sum of the previous two.) Demonstrate rows containing three numbers in arithmetical progression, i.e. differing by the same constant (there are infinitely many of these!) 7-21-35 with common difference 14, next 1001-2002-3003 and then 490314-817190-1144066. Interest note: There are no such triplets in either geometric or harmonic progression, see David Wells, Curious and Interesting Numbers, Penguin, 1987.

A STRUCTURED PROBLEM in DIOPHANTINE EQUATIONS

Posed by C.J. Roberts of Twyford, Berkshire.

Problem CJ_1. Find positive integers (A,B,C) which satisfy $A^2 + B^2 = C^3$.

(a) Investigate possible C in the range 2 to 100....2,5,8,10 etc.

(b) Identify any such C for which there are multiple solutions for (A,B)....5,10,13 etc.

(c) Propose the rule for C whch admits at least on solution (A,B).

(d) Propose the rule for C which admits more than one solution (A,B).

(e) Propose a rule which, given C, leads to the number of distinct solutions (A,B). Note that C = 65 or 85 has 8 solutions,

Fig 1 Binomial Coefficients: Sample Program

```
10    CLS
20    PRINT"*****************************************************"
30    PRINT"*                                                   *"
40    PRINT"*     EVALUATION OF BINOMIAL COEFFICIENTS PROGRAM    *"
50    PRINT"*                                                   *"
60    PRINT"*****************************************************"
70    LOCATE9,40:PRINT"n"
80    LOCATE10,26:PRINT"EVALUATION OF C":PRINT TAB(42);"r"
90    LOCATE15,32:INPUT"n=";N:LOCATE17,32:INPUT"r=";R
100   LET D=1:IF R=0 THEN GOTO 140
110      FOR Y=0 TO R-1
120      LET D=D*(N-Y)/(R-Y)
130      NEXT Y
140   LOCATE20,29:PRINT N:PRINT TAB(31);"C":PRINT TAB(32);R:LOCATE
      21,36:PRINT"=";D
150   LOCATE23,19:PRINT"PRESS (SPACE BAR) TO START AGAIN":H$=
      INPUT$(1):IF H$ =""THEN RUN
```

while C = 25 or 50 or 100 has three solutions.

C.J. has solved (c) & (d) but so far not (e) completely.

A TWO PERSON GAME — ODDS & EVENS

Posed by C.J. Roberts of Twyford, Berkshire.

Problem CJ_2. Playing alternately two players shall draw either 1,2 or 3 counters from an initial pile of 2N + 2 counters. Say 23 for example...the winner is the player who finishes with an even number of counters.

Construct a winning algorithm for play in this case. How does it vary when you can draw from 1 upto M counters at any given time?

Responses to both the Beginners Lesson Three, THE BINOMIAL COEFFICIENTS and/or any of the problems CJ_1, and CJ_2 may be sent to Mike Mudge, 22 Gors Fach, Pwll-Trap, St. Clears, Carmarthen, Dyfed SA33 4AQ, tel 01994 231121, to arrive by 1st July 1995. Any complete or partial solutions received will be judged using suitable subjective criteria, and a prize in the form of a £25 book token, or equivalent overseas voucher, will be awarded by Mike Mudge to the "best" solution arriving by the closing date. Such contributions should contain a brief description of the hardware used, details of coding, run-times and a summary of the results obtained; all in a form suitable for publication in *PCW*. Additionally, readers' comments upon the general or specific nature of this month's column would be most welcome. In particular, references to any recent work (either published or unpublished) on Diophantine Equations of the general type $a^n+b^n+c^n.... = A^m+B^m+C^m...$ would be appreciated.

Please note that material can only be returned if a suitable stamped addressed envelope is provided.

FEEDBACK — BOTH POSITIVE AND NEGATIVE

There have been a number of letters from readers who were awarded prizes "by *PCW*" rather than "by Mike Mudge", i.e. results announced before the July 1994 issue of *PCW*, and who have not yet received their prizes. These were genuinely earnt and awarded: please contact Adele Dyer at *PCW* to resolve this matter as soon as possible.

Readers who are experiencing difficulties in purchasing recent books in number theory — particular problems have arisen with the second edition of Richard K. Guy's Unsolved Problems in Number Theory — are advised to contact Hinchliffe Books at 15 Castle Street, Thornbury, Bristol BS12 1HA. Tel: Thornbury (01454) 415177.

Light relief! Are there only finitely many squares, apart from 10^{2n}, $4x10^{2n}$ and $9x10^{2n}$ with only two decimal digits? e.g. $38^2 = 1444$, $88^2 = 7744...3114^2 = 9696996...$?

Review of Numbers Count -139- November 1994: INT to ZIG-ZAG and something different

Despite its simple requirements in terms of logic and hardware, the FLOOR FUNCTION was totally ignored by readers' responses. Can someone please say why? Yes, I can! Apologies, but no proofs came from *PCW*.

$$"\left[n/p \right]+\left[n/p^2 \right]+\left[n/p^3 \right]+..$$

the sum being continued until the terms become 0, i.e. until p^r is greater than n." was intended. Now have a go at problems 0 & 1 for a future prize award, together with ZIG-ZAG.

However, Neil Croll's problem produced much interesting response; because of the nature of his work he is the very worthy prizewinner at 9 Carsington Crescent, Allestree, Derby DE22 2QY. Mention must be made of the link with "Pieces of Eight" from *PCW* 1992 February and the generalisation to an alphabet of k characters, say 0,1,2..k-1. What is the minimal length string that when viewed cyclically includes all possible length-r strings as substrings? Furthermore, how many such minimal length strings are there?

For k=2 see Martin Gardner's discussion of Bacon's Cipher in Knotted Doughnuts and other Mathematical Entertainments. The number of solutions is 2^x where $x=2^{r-1}-r$ so for Croll's original problem there are 2^{26}. Interested readers are also referred to The DIPOLE Column in IEE News (no. 28) of 3rd August 1989 and its follow-up.

PCW Contributions welcome

Mike Mudge welcomes readers' correspondence on any subject within the areas of number theory and computational mathematics, together with suggested subject areas and/or specific problems for future Numbers Count articles.