Fraction **action**

Mike Mudge presents continued fractions — when are they periodic, and how long are the periods?

Definition: an expression of the form $a_0+1/(a_1+1/(a_2+1/(a_3\ldots$ is called a **regular**, or **simple**, continued fraction. Throughout this work a_r will denote positive integers. (a_0 may be zero.) The SIMPLE continued fraction numerically equal to any rational number (i.e. the quotient of two integers) must terminate. That is, have only a finite number of partial quotients a_r,; although such expressions have certain applications, including the design of gear trains, they have very limited appeal in computational or pure mathematics. For example, 105/38 = 2 + 1/(1 + 1/(3 + 1/(4 + 1/2))). To simplify this somewhat cumbersome notation, we write 105/38 = (2;1,3,4,2.).

Theorem A. Look at *Continued Fractions* by A. Ya Khinchin (Phoenix Science Series, The University of Chicago Press, 1964). The necessary and sufficient condition for a simple continued fraction to be finite is that it represents a rational number.

Theorem B. *loc.cit.* above. The necessary and sufficient condition for a simple continued fraction to be **periodic** is that it should represent a **quadratic irrational**. That is, a non-integer real root of a quadratic equation: $ax^2 + bx + c = o$ where a, b and c are integers, a not equal to zero.

● **Problem 1.** Write a simple computer program to generate the (**finite**) continued fraction corresponding to any given positive rational number, i.e. input p/q and output $(a_0;a_1;a_2;a_3,\ldots a_n.)$.

● **Problem 2.** Write a simple computer program to solve **exactly** any given quadratic equation with integer coefficients, i.e. input a, b & c as in $ax^2 + bx + c = 0$ and output the roots as P$\pm$SQRT(Q).

It is suggested that the reader now experiments with simple periodic continued fractions such as (0;1,1,1,…), also (2;3,4,3,4,3,4,…) to see the quadratic equation whose root they represent. Note in the first example, x = 0+1/(1+x), while in the second example,
x - 2 = (0;3,4,3,4,3,4,...) = y
say where
y = 1/(3+1/(4+y)).
Hence, the desired quadratic equations and exact values for x & y can be found.

The more complicated experiment is to start with a given quadratic equation and determine the continued fraction expansion of any positive real roots which it may have. Note: these **must** be periodic; the analysis may be beyond the mathematical experience of some readers, but its omission does not affect the continuity of the rest of this discussion. Now restrict the quadratic equation to the form, $x^2 - a = 0$, and focus on the root SQRT(a). In their paper *Some Periodic Continued Fractions with Long Periods* (*Mathematics of Computation* vol 44, number 170, April 1985 pp 523-532), CD Patterson and HC Williams used The University of Manitoba Sieve Unit (UMSU), "a machine similar to DH Lehmer's DLS-127", to investigate cases of long periodicity. Theoretically, they identified four classes of 'a' of interest: (1) a $\equiv$ 3 (mod 8) 'a' prime; (2) a $\equiv$ 7 (mod 8) 'a' prime; (3) a $\equiv$ 6 (mod 8) 'a'/2 prime; and (4) a $\equiv$ 1 (mod 8) 'a' prime. Denoting the period by p(a), typical results in each of these classes are:

a 2186009851 2763423391 2340752254 18901431649
p(a) 151838 170804 157036 433383

● **Problem 3.** Attempt to determine the period of the simple continued fraction expansion of SQRT(a) in such a manner that the investigation can be extended to the orders of integers indicated above. Verify that the period is bounded by:
$f(a) = a^{1/2}\log\log(a)$ if a $\equiv$ 1(mod 8) and by
$f(a) = a^{1/2}\log\log(4a)$ otherwise.

● **Something different**

In March 1986, readers were invited to find integer solutions p,q,r,s,t for
5(p2+q2+r2+s2+t2)2 = 90pqrst + 7(p4+q4 +r4+s4+t4).

An extensive investigation by *PCW* reader, Duncan Moore, generalised the 90 to 5n and led to the following questions:
(a) Are there any solutions with three of p,q,r,s,t sharing one factor and the other two sharing a different factor? If not, then the search for solutions with three only sharing a common factor could be significantly speeded up.
(b) Are there any solutions with n = 1 or with n = -1?

Any investigations of the above problems, together with answers (either complete or partial) to Duncan Moore's questions, should be sent direct to: Mike Mudge, 22 Gors Fach, Pwll-Trap, Carmarthenshire SA33 4AQ (tel 01994 231121), to arrive by 1st December. The author also welcomes comments on the subject areas chosen this month: namely, continued fraction theory and Diophantine equations. Details of recent results either published or unpublished in these areas would be particularly appreciated.

Interesting Powers of Ten

Hugo Steinhaus' problem (*PCW,* January) was of great interest. This produced a very interesting set of responses. Worthy of mention in the Interesting Powers of Ten, are Paul Leyland's conclusion that there are no less than 1063017, other than those quoted — the result of almost three hours' computing time on a DEC Alpha. Nigel Hodges used Microsoft C++ on his Packard Bell up to 2^{10000} in three seconds and then established some associated probabilities. Steinhaus, being simple to comprehend, yielded a great deal of results. However, the clear prize-winner is Richard M Tobin, 2 Flr, 53 Spottiswoode Street, Edinburgh, EH9 1DQ, who programmed a Sun Sparcstation 5/110 in C and summarised all of the Steinhaus cycles up to and including twenty fifth powers! This latter took eight days and revealed nine perfect digital invariants (including 1), the next one having 24 digits.

PCW *Contributions Welcome*

Mike Mudge welcomes correspondence from readers on any subject within the areas of number theory and computational mathematics, together with suggested subject areas and/or specific problems for future *Numbers Count* articles.